latter might well include reports on special methods of problem solutions, pedagogical discussions, accounts of original investigations, the organization of a particular topic or chapter in some collegiate subject, etc. That such a combined program has elements of strength greater than one composed either entirely of selected papers on specified topics or papers entirely unrestricted, seems unquestionably answered in the affirmative by the result at Chicago. In anticipation of similar programs at future meetings, the officers of the Association earnestly urge this opportunity upon the attention of our members, so that when a call comes for either type of papers, there may be a ready and hearty response. The "Questions and Discussions" in recent numbers of the MONTHLY indicate that our members appreciate an open forum, and it seems evident that a fuller opportunity should be provided than the crowded pages of the MONTHLY allow.

On Thursday evening the joint dinner of the Society and the Association was held at the Quadrangle Club with an attendance of 73. The toastmaster, President Cajori, called upon the following members, who responded with brief remarks: Professor E. H. Moore, University of Chicago; Professor W. B. Ford, University of Michigan; Professor E. V. Huntington, Harvard University; Professor L. E. Dickson, University of Chicago; Professor E. L. Dodd, University of Texas; and Professor E. R. Hedrick, University of Missouri. Much interest was expressed in various phases of war activity and in the share which the Association's members have in the country's service. By request, the secretary of the Chicago Section of the Society and the secretary of the Association collated rather hastily and read a list of 48 American mathematicians known to be in national service at present. About forty members took advantage of the invitation of the American Association of University Professors to attend the dinner on Friday evening at the Quadrangle Club. As in the case of earlier meetings, so at Chicago admirable accommodations and plans made by the local members of the Committee on Arrangements helped greatly to further social intercourse and to render everyone's enjoyment the more complete. The members of this Committee and those of the Program Committee merit much praise for the thoughtful planning which made the meetings a success.

President Cajori presided except on Thursday afternoon, when he called Professor E. W. Davis to the chair. Professor L. E. Dickson, president of the American Mathematical Society, presided at the joint session on Friday afternoon. At this session Professor W. B. Ford gave his retiring address as chairman of the Chicago Section of the Society, on the subject "A conspectus of the modern theory of divergent series." On behalf of the Association, Professor L. D. Ames of the University of Missouri gave an address "On a definition of the real number system by means of infinite decimals." Professor Ford's paper has been offered to the *Bulletin* of the Society for publication. The following is an abstract of Professor Ames's paper:

The purpose of this paper was to present a definition of the real number system which, while sound logically, shall be pedagogically simpler than the

well-known methods. Pedagogically the number system is built up by successive steps; with one exception each step follows rather closely our intuitive notions; but in the definition of irrational numbers all the commonly accepted methods of approach break rather sharply with our intuitions and are rather complicated. The mode of procedure suggested in this paper is as follows: It is first shown that all rational numbers can be expressed as terminating or repeating decimals, and that conversely all terminating or repeating decimals are rational. Then examples of individual infinite decimals which do not repeat are given; these are not rational, but they make as vivid an appeal to the student as do the rationals. A real number is then defined as an expression in the decimal notation, this being justified pedagogically and logically. A logical treatment of the properties of real numbers follows closely the natural suggestions of the decimal notation. The notion of a continuum is approached as follows: The rational number system has certain properties of order which are carefully stated. The student knows some infinite decimals which are not rational; these can be inserted one at a time between rationals, and the enlarged system will have at each stage the properties just mentioned. To speak crudely, there seem to be vacant spaces where new individual numbers can be inserted between those already given. On the other hand it is not possible to insert in the set of all infinite decimals a single new element so that the enlarged set shall have the properties stated. To speak crudely again, all the vacant spaces seem to be filled. This idea is the basis of the definition given of a continuum.

The other parts of the program are grouped under three leading heads, and abstracts are given corresponding to the numbers in each list, together with reports of some further informal discussions.

Contributed Papers.

(1) "The Graph of $f(x)$ in Line-Coördinates for Complex Numbers." Professor A. F. Frumveller, Marquette University.

(2) "Note on the Generalization of the Witch and the Cissoid." Professor F. H. Hodge, Franklin College.

(3) "Fermat's Method of Infinite Descent." Professor W. H. Bussey, University of Minnesota.

(4) "On the Disciplinary and Applied Values of Mathematical Study." Professor C. N. Moore, University of Cincinnati.

(5) "On the Content of a Second Course in Calculus." Professor E. J. Moulton, Northwestern University.

(6) "Comments on Mathematics in the High School." Professor Harris Hancock, University of Cincinnati.

(1) The graph of $f(x)$ in ordinary point-coördinates leads to a 3-dimensional diagram when x or y is taken as complex; and to a 4-dimensional diagram when both are complex. Professor Frumveller's paper was described as an attempt to escape from this difficulty by employing line-coördinates. A base-line

OO_1 and two lines, OX, OY, perpendicular to it and in one plane, are the framework used. The "locus" of the moving line is that part of the plane over which the line is free to range. Lacunary regions exist, bounded by the "envelope" of the line.

When x, or y, or both, become complex, the lines OX, OY are replaced by planes. The moving line now ranges over a 3-space, and generates ruled surfaces, the lacunary regions being now bounded by the curves called respectively the "edge of regression," and the "line of striction." A method of handling this diagram conveniently was indicated.

(2) In this note Professor Hodge presented two simple constructions, each leading to a family of quartic curves involving a parameter k. As k becomes infinite the first construction merges into the classic construction for the witch and the equation becomes the equation of the witch. Under the same conditions the second construction becomes the construction for the cissoid and its equation becomes the equation of the cissoid.

(3) Professor Bussey's introduction to Fermat's famous method of infinite descent and discussion of its range of usefulness will appear in full in a forthcoming issue of the MONTHLY.

(4) In this paper the disciplinary value of mathematical study as a training in reasoning was exhibited by Professor Moore in the following fashion: Certain well-known examples of deductive reasoning in the fields of public debate and natural science are analyzed into their logical elements. Parallel with each example are given one or more examples of deductive reasoning in elementary algebra and geometry, which are built up from precisely the same logical elements. Thus the medium for the transfer of training is shown in explicit form. The applied value of mathematical study is shown by pointing out that a knowledge of elementary mathematics is essential for a proper understanding of the scientific discussion of almost any subject that can be studied scientifically. Thus, for example, some knowledge of mathematical notation and procedure is necessary, not only in all the natural sciences, but also in psychology and the social sciences.

(5) In his discussion Professor Moulton commented on the obvious general need of a second course in calculus, and on the diversity of ways in which the need is met. One source of difficulty in outlining a course for general consideration lies in the differences in the preparation of students in various institutions due mainly to the different lengths of the first course in calculus. An outline was presented of a course to follow a first course of about a hundred lectures. It is the hope of the editors that a fuller account of this paper may be given in an early number of the MONTHLY.

(6) Professor Hancock said that the arguments given in recent years by various non-mathematicians in support of the classics and English apply with equal force to mathematics. Noteworthy are a book on "The value of the

classics" by Dean Andrew F. West of Princeton University, a speech by Dean Roscoe Pound of Harvard Law School, the speech by Ex-President Grover Cleveland at the inauguration of Woodrow Wilson as president of Princeton University, and the answering comments of President Wilson. Professor Hancock urged also the desirability of having an experimental high school which shall be entirely antipodal to the Flexner "Modern School," and reported on various investigations of the past few years, chiefly of one in Cincinnati in which business men and a general group of college and university teachers voted predominatingly for a rigorous course which should include mathematics.

SESSION ON DESCRIPTIVE GEOMETRY.

(1) Address: "Descriptive Geometry and its Merits as a Collegiate as well as an Engineering Subject." PROFESSOR W. H. ROEVER, Washington University.

Discussions by

(2) PROFESSOR F. HIGBEE, Department of Descriptive Geometry and Drawing, State University of Iowa.

(3) PROFESSOR A. V. MILLAR, Department of Drawing, University of Wisconsin.

(4) PROFESSOR ARNOLD EMCH, Department of Mathematics, University of Illinois.

(5) MR. WILLARD W. ERMELING, Instructor in Descriptive Geometry, Crane Junior College, Chicago.

(6) MR. W. F. WILLARD, Instructor in Drawing, Carl Schurz High School, Chicago.

General Discussion.

(1) In his address Professor Roever traced the development from the early attempts to solve the problems encountered in building and stone-cutting to the descriptive geometry of Monge, and showed how the needs of the artist were responsible for perspective, which in turn brought forth projective geometry, axonometry and photogrammetry and thus completed the work begun by Monge. He emphasized the two purposes: (1) to represent, by figures which lie in a plane, the objects of space; (2) to solve, by geometric constructions which can be executed in a plane, the problems of space; and then showed that the Mongean method fulfills best the second purpose and perspective the first, but that the axonometric method of making parallel projections satisfies simultaneously both. He called attention to the fact that the development of descriptive geometry in Europe has been confined almost exclusively to the technical schools and that in this country also it is confined to the engineering schools, being taught there to a limited extent only. He closed by saying that the hope for its development in this country depends on its introduction into the colleges and universities, that for this there is sufficient justification, that the requirements demanded of teachers of descriptive geometry should be raised and that mathematicians should familiarize themselves with it more fully.

Professor Roever's address will appear in full in an early number of the MONTHLY.

(2) "The study of descriptive geometry," writes August Comte, "possesses an important philosophical peculiarity, quite independent of its high industrial utility. This is the advantage which it so preëminently offers in habituating the mind to consider very complicated geometrical combinations in space, and to follow with precision their continual correspondence with the figures which are actually traced—of exercising to the utmost, in the most certain and precise manner, that important faculty of the human mind which is properly called imagination, and which consists, in its elementary and positive acceptation, in representing to ourselves, clearly and easily, a vast and variable collection of ideal objects, as if they were really before us. . . ." The more often one reads this statement and considers its true meaning, the more one becomes convinced that it contains the first and last word on the value of descriptive geometry as a college subject.

This "high industrial utility" of which Comte writes has quite generally been conceded. As evidence of this fact one might point out that from the beginning of technical schools down to the present descriptive geometry has held an important place in their courses of study, and has held it, one is tempted to add, in spite of the revolutions through which these same courses of study have passed with changing administrations and times. The relation which exists between descriptive geometry and designing is fundamental; it is the bedrock upon which may be based a knowledge of the art of drafting. As a fundamental part of the training of a designer, descriptive geometry should aim to accomplish three things: First, it should teach orthographic projection, and teach it most thoroughly. A designer must not only be able to make drawings but he must read them as well, and orthographic projection may truly be called the grammar of that language which is the means of communication between designers and builders. Secondly, descriptive geometry should develop the ability to solve graphically problems concerning the relations of points, lines and planes. These are, of course, but the elements in the representation of all engineering structures and it is essential that designers and draftsmen be trained to solve problems relating to them directly on the drawing-board. Thirdly, and perhaps most important of all, descriptive geometry should promote the ability to analyze a problem into its component parts, to reason logically and clearly from a given set of conditions to a required set of conclusions, to build up from a drawing a mental picture of what is there represented, or the reverse of this process; for without the ability to analyze, to reason, to visualize, and to translate from one language to another, a draftsman is lacking in the essential qualifications of his calling.

But since descriptive geometry has become established as an essential part of a technical course of study, perhaps we may assume its value in such a course and pass on to consider, without further comment here, whether or not the

subject might with profit be seriously considered as deserving of a place in a non-technical college course. Let us turn again to our quotation from Comte. From this it must be evident that descriptive geometry, quite apart from its recognized practical value, is entitled to serious consideration for its educational advantages. If it has such advantages, then undoubtedly the subject deserves a place in a non-technical course of study.

Being a form of geometry, it should need no defense before an association such as this. If mathematics in any form possesses educational value, apart from the unquestioned usefulness of such subjects, then it should be agreed that descriptive geometry likewise has an educational value and is therefore deserving of recognition as a proper subject for study. But in addition to this perhaps recognized fact, it should be kept in mind that in descriptive geometry analysis and logical deduction are verified by graphical solution. Thus, not only does descriptive geometry possess all the merits of other forms of geometry but in addition it insists upon a mental attitude which leads one to visualize the problem, to see it in all its parts, and to work with conditions which are actually represented by a drawing. "A mathematical problem," to quote from another writer on the subject, "may usually be attacked by what is termed in military parlance the method of systematic approach; that is to say, its solution may be gradually felt for, even though successive steps leading to that solution cannot be clearly foreseen. But a descriptive geometry problem must be seen through and through before it can be attempted. The entire scope of its conditions, as well as each step toward its solution, must be grasped by the imagination. It must be taken by assault." Even to the sceptic it must be obvious that such a form of mental training is valuable; and when to this is added the usefulness of descriptive geometry as a real means of solving difficult problems with a considerable degree of precision, it should be granted that the subject might properly be placed in every college course.

(3) In descriptive geometry, objects of three dimensions are represented in outline on a plane sheet. This method of conveying an idea from one person to another by means of a drawing is widely used and is one of the most accurate and concise methods employed for that purpose. It is often superior to a written or oral description. The fundamental principles involved in making such a representation of an object are the principles included in descriptive geometry. The subject therefore has great practical importance and the principles are so universally employed that many times the user does not realize he is employing descriptive geometry. Those who have done much designing, either of buildings or of machines, and who have not made a formal study of descriptive geometry, use its principles; and if they take up the study of the subject later they find it very easy. For those who are preparing themselves to do design work, the study of the subject develops the reasoning and visualizing powers and enables them to progress more rapidly and intelligently with design.

The visualizing process, so often spoken of in connection with descriptive

geometry, depends upon reasoning. The student reasons from the given conditions that the objects or magnitudes must be in a certain position in space and by his reasoning power builds up a mental picture, the clearness of which depends upon the clearness of his reasoning. The subject, therefore, in addition to its practical application, gives excellent training in reasoning—a particular kind of reasoning which results in a definite mental picture. Students who have had thorough training in mathematics, particularly calculus, find descriptive geometry much easier than those who have not had such training. The subject is peculiar in that the problems are usually seen as a whole. The student first analyzes the problem, which involves seeing it in complete form and as well the various fundamental steps which must be taken to reach the conclusion, and then he makes the construction which involves taking these steps in order until the conclusion is reached. There can be no better training given than by this process of grasping the problem as a whole and then testing the accuracy of the conclusion by an actual mechanical construction.

There are only a few fundamental principles in the subject, but these principles must be thoroughly learned if the student is to be master of the subject. He must know these principles as he knows the fundamental operations in arithmetic if he expects to solve successfully original problems. The fact that there are few fundamental principles and that these can be thoroughly mastered makes the subject just that much more valuable. With the laboratory practice which usually accompanies descriptive geometry comes the value of motor training and the emphasizing of principles studied in the class-room. This part of the subject should by no means be omitted and a fair degree of accuracy should be demanded. If the students in our colleges are to become proficient in the use of English, instructors in all subjects must help. Frequently a student must be told that he appears to understand the problem under discussion but that his statements are inaccurate and would be unintelligible to a person not familiar with the subject. He must repeat one or more times and may need the help of other students before the answer is satisfactory. There is no better subject in the college curriculum for emphasizing the need of clear and concise English than descriptive geometry.

The subject then is not merely of practical value but gives also excellent opportunity for training in reasoning, thoroughness, motor activity, and English.

(4) Professor Emch found himself in agreement with most of the views and arguments advanced by Professor Roever. A course in descriptive geometry, or constructive geometry, is certainly of great benefit to the student of mathematics. As in other branches of science, it is desirable that the student should first acquire a certain amount of geometric knowledge of forms and their representation before he is asked to argue about the logical foundations of science. This principle should be observed earlier in the teaching of geometry in the high school, *i. e.*, much more attention should be paid to the constructive side of geometrical instruction in its elementary stages. Much complaint about poor

results in many classes in geometry is probably due to this neglect. At the University of Illinois this want is met by a course in constructive geometry, which is open to students who have had analytic geometry. As a rule this course is very well attended; it serves as an introduction to projective geometry and other more advanced courses in geometry. It comprises chapters on the various possibilities of geometric construction, accuracy and simplicity of construction, inversion in the plane and in space (including stereographic projection), isometric projection, orthographic projection, perspective, the analytic form and mathematical principles of these projections, collineation, cross-ratio and its invariance in projection, elementary synthetic theory of conics and quadrics, description of curves by linkages.

It would seem that a course of this or a similar sort, as suggested by Professor Roever, should be taken by every prospective teacher of mathematics or writer of mathematical text-books along geometric lines. The great number of faulty drawings in many text-books on geometry would thereby be avoided. Moreover, no course can be made more inspiring for future geometrical research than constructive, or descriptive, geometry for beginners who intend to specialize in mathematics.

(5) Professor Roever's address suggests to Mr. Ermeling several questions to which attention may well be drawn. Perhaps the first that occurs is this: Should the subject of descriptive geometry be confined to technical schools? Prof. T. E. French of Ohio State University has said that descriptive geometry has three important functions: (1) to train the student to think in terms of three dimensions; (2) to visualize quickly and accurately; (3) to build up a clear mental image by a training in constructive imagination. These are of primary importance to the engineer, who deals with three-dimensional things. Are they any less important to the general student? Is not one of the needs of a course in mathematics this very training in visualization? We have all seen the pleasure of the students of analytical geometry in making graphs of their work. Are we doing our duty by the student of mathematics if we give him the symbols, the values, the equations and omit the other benefits so easily obtained by a course of mathematics worked out over the drawing-board? Is it not like trying to teach chemistry or physics without laboratory work?

It is undoubtedly true as Prof. V. T. Wilson says in the preface to his text-book, "that in the application of the principles of descriptive geometry to the making of working drawings, the necessary modifications have often caused the loss of sight of the fact that descriptive geometry is a mathematical subject." This raises the questions of (*a*) what should be the qualifications of the teacher of descriptive geometry, and (*b*) whether the subject should be taught by the department of drawing or by the department of mathematics. A little consideration would probably lead us to see that the teacher of the subject should have not only a mathematical training, but also a practical experience in the drafting room; or, to state it more broadly, not only engineering training but

engineering experience. It does not then greatly matter whether the subject be considered a part of the drawing department or of the department of mathematics. The important thing is to have descriptive geometry related directly to the other branches of mathematics.

With all the foregoing facts in mind, it would seem that descriptive geometry has a value such that the subject could well be required in a general collegiate training in mathematics and not confined to the engineering courses as at present.

(6) In attempting to discuss the merits of Professor Roever's excellent paper, it might be well to answer first some of the questions which he submitted. (1) It is Mr. Willard's opinion that the preparation of a teacher of descriptive geometry should include an engineering education, inasmuch as the subject finds its greatest usefulness in that field. If a student majors in mathematics in a purely academic college course, he should likewise have a course in descriptive geometry. (2) It would seem from the meager data available that most of the smaller colleges present the subject in connection with the mathematics department, whereas, in the engineering schools, it is presented in conjunction with the drawing department. This is so because engineers emphasize the drawing phase for its practical application to many problems in industrial life. The colleges do not emphasize the drawing part of the subject perhaps because the department of engineering has not been established or developed, the student on that account giving the analytical part of the subject his exclusive attention. (3) The old texts, which have become fossilized, are in a measure responsible for a great deal of the distaste which students have for the subject. In the speaker's experience, where the subject was presented in the freshman year of an engineering course in college, many students were predisposed to failure, through the reputation of descriptive geometry which was passed on by those who had completed the course, a majority of these students having come from schools where not all the shops and drawing departments were yet established.

The advisability of extending this subject into the secondary schools rests largely with the school itself. It is obviously undesirable to require boys and girls alike to take descriptive geometry in a purely academic or college preparatory course in high school; but, where our city schools have included technical courses with a view to engineering preparation, it is quite possible for such students to grasp the subject, and highly desirable as well, especially in the third or fourth year. It is also well to observe that boys who have had a thorough course in plane and solid geometry have very little difficulty in descriptive geometry. In the Chicago high schools, technical course students are concerned in their sophomore year exclusively with developments and penetrations of solids. In these exercises are many problems involving the theorems of descriptive geometry. Why would it not be easy for these boys to cover a satisfactory course in descriptive drawing immediately following their sophomore year? Our universities and engineering schools, as a rule, are loath to accept the work of the secondary schools for entrance in this particular, and perhaps justly so, because of the

varying success of our high schools in carrying on this work. Boys who leave from our best technical high schools for the universities and engineering schools very often are required to duplicate this work, much to their chagrin. It might be well for our higher institutions, through their examiners, to differentiate such institutions as are proficient and do offer a course in descriptive drawing, and credit this toward advancement in their college course. When this is done, there will certainly exist a happier relationship between our secondary schools and universities.

In the ensuing general discussion, Professor Van der Vries and Mr. Ermeling brought out the fact by a show of hands that in most of the institutions represented at the meeting the subject of descriptive geometry is taught not in the department of mathematics but in the department of drawing, and by technically trained instructors, although Mr. Willard and Professor Archibald were inclined to believe that a fuller investigation would show the contrary to be the case in the country at large. Professor E. W. Davis described the method as carried out at the University of Turin, where figures are sketched upon the blackboard in free-hand; he stressed the great help which descriptive geometry affords the students of solid geometry and on this ground urged this or the study of perspective as a part of the regular course in high-school geometry.

As an instance where the methods of descriptive geometry lend themselves readily to the satisfactory solution of a problem often treated unscientifically even by mathematically trained persons, Professor Roever described the method for drawing the representation of a line perpendicular to a given plane. A lively interchange of views between Professors Roever and Millar brought out the fact that the principles of representation (*e. g.*, in isometric projection, which was at the time under discussion) are not always applied consistently. In response to Professor Millar's doubt as to the ability of high-school pupils to handle descriptive geometry readily, Mr. Willard expressed his decided judgment based on his teaching experience that they do show a sufficient ability. Professors Dodd and Hadley pointed out that the course is sometimes given successfully, paying only secondary consideration to the extreme accuracy of the constructions. Following Professor Burgess's remark as to his unhappy memory of the course made up as it was of a long succession of puzzling drawings carrying out meaningless rules, Professor Wilczynski summed up the discussion by saying that it is altogether evident that there has been too much purely mechanical following of particular methods without a proper presentation of the mathematical principles involved, and that the leading colleges and universities must do their share in bettering this instruction and making more certain the undoubted benefits which come from the study of descriptive geometry.

Reports of Standing Committees.

(1) National Committee on Mathematical Requirements.

(*a*) "Scientific Investigations of the Committee." Professor A. R. Crathorne, University of Illinois.

(*b*) "The Work of a Committee representing the Central Association of Science and Mathematics Teachers." MR. J. A. FOBERG, Crane Junior College, Chicago, Chairman Coöperating Committee.

(2) Committee on Libraries.

(A report of this committee was published in the October MONTHLY in the form of a list of 160 mathematical books for schools and colleges.)

Discussion opened by PROFESSOR H. E. SLAUGHT, University of Chicago.

(3) Committee on Mathematical Dictionary. Preliminary Report by the Chairman, PROFESSOR E. R. HEDRICK, University of Missouri.

(4) Committee on Annals of Mathematics. Report by the Chairman, PROFESSOR E. H. MOORE, University of Chicago.

(5) Committee on Bureau of Information. Report by the Chairman, PROFESSOR J. B. SHAW, University of Illinois.

(1*a*) The subcommittee on scientific investigations is at present studying the application of the theory of correlation to educational data. Several thousand complete records of high-school pupils in different parts of the country are being collected to form a statistical basis for the investigation. The correlation coefficients not only for mathematics and other subjects, but, for comparison purposes, the coefficients for many other pairs of subjects will be discussed. The question to be answered is this: "If subject A is considered by educators to be a fundamental part of all secondary curricula, does proficiency in mathematics count more or less than proficiency, say, in stenography, or civics, towards an increase of proficiency in A?"

There is a fairly well-defined theory believed in by most psychologists to the effect that training gained in the study of one subject is of use in the study of some other subjects. The technical word "transfer" is used in this connection. The real question at issue in connection with formal discipline is the amount and method of transfer. A commonly accepted statement of this principle of formal discipline (Bagley's) is as follows:

> "The present interpretation of the doctrine of formal discipline is based upon the belief that specific habits may be generalized into ideals and prejudices, which in turn make possible the acquisition of similar habits in new fields, as when from the specific habits of accuracy and close reasoning developed in the school exercises in mathematics one comes gradually to idealize accuracy and close thinking as methods of procedure that will bring desirable results in other fields."

One of the first steps in answering the question "Should mathematics be required of all high-school pupils," would be then to investigate the "spread," "transfer" or "generalizing power" of mathematics. We all believe that a boy or a girl who has taken a thorough course in algebra will do some things better than one who has not taken such a course. What these things are and how much better the first pupil will do them are things to be found out. The number which represents the "how much better" we might call the "factor of transfer" or "coefficient of spread." The object of this subcommittee is to find out as far as possible the connection between this "factor of transfer" and the correla-

tion coefficient. The object is not primarily to find arguments in defense of mathematics, but an open-minded investigation of correlation theory applied to educational measurements, with enough data behind it to make the results conclusive.

The Council of the Mathematical Association has voted the sum of one hundred dollars to take care of the great amount of purely clerical work involved in gathering the records used in this investigation.

(1*b*) At the Chicago meeting of 1916 the Mathematics Section of the Central Association of Science and Mathematics Teachers ordered the appointment of a committee to coöperate with the National Committee on Mathematical Requirements. Professor W. W. Hart, of the University of Wisconsin, then chairman of the Section, appointed a committee of twelve, distributed over the territory covered by the Association, and during the year past this committee has acted in cooperation with the National Committee.

At the request of Professor J. W. Young, a report was prepared by a subcommittee, with Mr. Alfred Davis of the Francis W. Parker School as chairman, on the topic "Valid Aims of Mathematics Teaching in the Secondary School." This was one of several preliminary reports submitted to Professor Young, who is soon to publish a final report on this topic. This subcommittee report[1] will appear in *School Science and Mathematics*, beginning in January, 1918.

A questionnaire addressed to prominent business folk and professional people of Chicago was prepared by another subcommittee, with the idea of investigating the opinions held by such persons concerning the value of mathematics teaching in the high school. This questionnaire was published at length in *School and Society* of November 17, 1917; discussion and comment concerning it has appeared in subsequent numbers.

Another subcommittee prepared a Report on First-Year Mathematics for the high school—a topic to be worked over more in detail in coöperation with the representative of the Association of Teachers of Mathematics in New England during the coming year.

After submitting its report at the Columbus meeting of the Central Association, the Coöperating Committee was continued for a year, with instructions to carry on the work in cooperation with the National Committee.

In the discussion which followed these two reports, Professor Hedrick, as the one who had been responsible in 1916 for the appointment of the National Committee, stated that it was distinctly understood that this Committee on Mathematical Requirements was to enlist the coöperation of committees of secondary-school teachers all over the country, and that these would be thus grouped together in effective service through this agency of the National Association. The whole discussion and recent correspondence makes manifest the growing consciousness over the country that here is for the first time in the history

[1] Reprints of this report and the questionnaire may be had from Mr. Alfred Davis, Parker School, 330 Webster Ave., Chicago, at ten cents per copy.

of American mathematics a powerful organized body which is investigating the whole question of mathematics as a school subject, employing in this study the definite information and the scientific training of the mathematicians of the whole country and at the same time putting aside natural prejudices in favor of mathematics as fully and honestly as is humanly possible.

(2) Professor Slaught, in speaking of the work of the Library Committee, first endeavored to show its proper setting in the general scheme of activities of the Association. Keeping in mind the general purpose of the Association, namely, to advance the interests of mathematics in the collegiate field, he mentioned three great streams of influence: (1) Meetings—national, sectional, and local; (2) Committees—on mathematical requirements, on libraries, and on bureau of information; (3) Publications—the MONTHLY, the *Annals*, and the proposed Dictionary. The national meetings, at best, can bring together only a small fraction of the members; but we may hope to see the sections become so numerous that every member may attend at least one meeting a year; and there is wide range and great opportunity for local clubs in the larger cities and in normal schools and colleges. The standing committees of the Association constitute the laboratories from which, as time goes on, will emanate the deliberative pronouncements of this national body on the great questions of every character which concern the vital interests of undergraduate mathematics in America. Already, our various committees are doing foundation work, especially in the three lines mentioned above. The publications of the Association are the channels through which must flow all these streams of influence to the individuals and institutions in the membership of the Association, and by means of which the interchange of ideas necessary to progress and growth is made. The MONTHLY, in this respect, stands in the most intimate relation to every member; the *Annals*, in its relationship to the Association, supplies a need that has long been felt in this country; and the proposed Dictionary opens up a great field of possible service to the college teachers and students of mathematics.

The Library Committee consisting of Florian Cajori, E. S. Crawley, Solomon Lefschetz, W. R. Longley, R. E. Root, and W. B. Ford, Chairman, was appointed in reponse to a pressing need based upon requests from many sources for assistance both in selecting mathematical books for a small college library and in convincing local authorities of the need of such books. The committee wisely limited this list to what would seem to be an irreducible minimum, and also classified the books according to the needs of students at the various stages of progress in the collegiate course.[1] The members of the committee have expended much time and energy in selecting these titles. They make no claim of infallible wisdom, in including this particular book or excluding that one, but it is clear that, in general, they have shown excellent judgment, and that they certainly have made a most worthy first approximation. Any college library which contains this

[1] The report was published in full in this MONTHLY for October, 1917, pp. 368–376. Reprints may be had from the Secretary, Professor W. D. Cairns, 27 King Street, Oberlin, O., upon request accompanied with a three-cent stamp.

list of books will provide suitable opportunity for its mathematical students who wish to know something more of the various subjects than can be found in the text-books used in the class-room. The committee will welcome suggestions of any sort with respect to this list. At the business meeting in the afternoon, the committee was requested by formal vote to prepare a list of text-books available for the various collegiate subjects, giving the publishers, the cost, and brief comments as to the character and scope of each. Presumably, the committee will include in this list all of the subjects mentioned in the tentative report of the Dictionary Committee immediately following this report. Such a list of text-books will be of the greatest service to collegiate teachers everywhere.

(3) Your Committee on Mathematical Dictionary, consisting of R. C. Archibald, H. L. Rietz, H. E. Slaught, D. E. Smith, and E. R. Hedrick, chairman, appointed at the Cleveland meeting, begs to report as follows:

After considerable correspondence and discussion at the Chicago meeting, the committee is agreed tentatively upon several important points, but none of these is irrevocably settled, and it is hoped that discussion here and suggestions by individuals later, either orally or preferably in writing, will assist the committee in revising and completing these tentative conclusions.

It is thought wise not to attempt any publication during the period of the war, but since the time required for the preparation of any such work would be very great, it is felt that we need not hesitate to initiate the work as soon as is practicable.

After some rough computation, we believe that a dictionary containing brief definitions of the words employed up to and including the end of collegiate work proper, or possibly including the first graduate year, could be condensed into two reasonable volumes which might be sold at approximately five dollars each. It is desirable that the committee have opinions as to the salableness of a work of this character and price, and as to its usefulness as thus limited in scope, and members are asked to advise the committee in writing.

As to the scope of the work, it is thought impracticable to include all mathematical terms used in the entire literature. Some line of demarkation seems inevitable. The committee has tentatively placed this line at the conclusion of the collegiate curriculum; that is, to include all words used in mathematical courses usually offered to undergraduates, or in some institutions to first-year graduates, and words that might occur in their assigned reading. This would cover the texts and references used in at least the following subjects: elementary algebra and geometry, trigonometry, plane and solid analytic geometry, and a first and second course in the calculus, together with elementary first courses in differential equations, higher analytic geometry (including curves and surfaces), higher algebra (including theory of equations, determinants, etc.), the theory of numbers, the theory of functions of real and of complex variables, the theory of groups, descriptive geometry, projective geometry, line geometry, differential geometry, mechanics, and the theory of probability and statistics.

It is assumed that no lengthy articles would be included, but only brief definitions, accompanied by necessary references to more complete sources, the ordinary linguistic comments, and possibly the equivalent words in French, German, and Italian.

A question exists regarding the inclusion of historical matter, since this is already reasonably covered by existing encyclopædias and by histories of mathematics. In certain cases, as in the entry "Taylor's Theorem," brief historical data might be given.

It was upon the basis of such a plan that the above estimate of size and price was made. Any enlargement of scope would entail obvious increase in size, time required for preparation, and price, and consequent decrease in probability of success in selling the dictionary widely to those who most need it.

It is felt that the venture is worth while only if it can be completed in a thoroughly scholarly manner, and so as to reflect credit upon the Association and upon American mathematics as a whole. To this end the interest and coöperation of some of our foremost scholars would be an essential, and we should hesitate to enter upon the work without the most certain assurance of such coöperation and support.

After hearing discussion here, and receiving suggestions from members personally, as a result both of this meeting and of the publication of this tentative report, the committee will try to formulate a more detailed and final report to be presented at some meeting of the Association during the ensuing year.

The above report was discussed briefly at the close of the business meeting in the afternoon. Two points were brought out clearly and evidently endorsed by the sentiment of those present: (1) That such a dictionary is greatly needed by those who are not within reach of large mathematical libraries, as is the case with most teachers and students of mathematics in the colleges of the country. This need was emphasized in most convincing manner by Professor A. F. Frumveller of Marquette University, Milwaukee, Wisconsin; (2) that such a dictionary is not greatly needed by the professors and research students of the graduate schools, and that the only justification for the Association's proposed sponsoring of such a venture is the real service which it may render in the collegiate field where at present no reference books comparable in scope and potential usefulness are available. This point of view was emphasized by Professor Slaught, who declared that his interest in the dictionary project was based entirely upon its adaptation to the needs of college teachers and students.

(4) A report of the Committee on the relations of the Association with the *Annals of Mathematics* was made by the chairman, Professor E. H. Moore, at the dinner on Thursday evening. Attention was called to the fact that the articles of expository or historical character for which the Association is supplying a subvention are now appearing in the *Annals* and that the next one will be by Professor D. R. Curtiss in the March issue on "A Chapter in the Theory of Equations." An appeal was made for contributions of articles of the character

in question, the present scarcity being attributed to the fact that this opportunity for the publication of such articles has so recently become effective. The committee will welcome any information concerning possible articles, especially those in which the exposition is made in the simplest and most direct form regardless of the space required in order to achieve this end.

(5) The Bureau of Information desires to report that it has served its purpose to some extent during the past year. It has replied to thirteen inquiries, furnishing information as to text books, history of mathematics, pedagogy of mathematics, a few problems of special interest to the proposer, and has thrown light on certain questions that puzzled the inquirers.

The chairman of the bureau, Professor J. B. Shaw of the University of Illinois, would offer the suggestion that the bureau be rather steadily advertised, as it probably has not yet come to the attention of many persons that such a bureau exists and is doing business. If the number of inquiries was greater the bureau would be able to maintain at least a page of replies in each number of the MONTHLY and it would be then advertised as a matter of course.

MEETINGS OF THE COUNCIL OF THE ASSOCIATION.

The Council held meetings on Thursday afternoon and on Friday and Saturday, ten members being present at each session.

(1) The following twenty-four persons and three institutions, on applications duly certified, were elected to membership:

To individual membership:

C. C. BRAMBLE, instructor in mathematics, U. S. Naval Academy, Annapolis, Md.
A. L. CANDY, professor of pure mathematics, University of Nebraska, Lincoln, Neb.
R. C. COLWELL, professor of mathematics, Geneva College, Beaver Falls, Pa.
C. G. CROOKS, professor of mathematics, Centre College, Danville, Ky.
P. J. DANIELL, assistant professor of applied mathematics, Rice Institute, Houston, Tex.
ALEXANDER DILLINGHAM, instructor in mathematics, U. S. Naval Academy, Annapolis, Md.
C. S. DOAN, head of mathematics department, Friends' Select School, Philadelphia, Pa.
THEODORE DOLL, fellow in mathematics, Northwestern University, Evanston, Ill.
H. H. DOWNING, assistant professor of mathematics, University of Kentucky, Lexington, Ky.
E. T. FRANKEL, statistician, Police Department, New York, N. Y.
L. M. GRAD, New York, N. Y.
W. H. HAYS, teacher, Broadway High School, Seattle, Wash.
L. S. JOHNSTON, professor of mathematics, State Normal School, La Crosse, Wis.
J. E. MCATEE, instructor in mathematics, University of Illinois, Urbana, Ill.

V. S. MALLORY, vice-principal and instructor in mathematics, High School, Dumont, N. J.

I. L. MILLER, professor of mathematics, Carthage College, Carthage, Ill.

M. J. NEWELL, Township High School, Evanston, Ill.

JESSIE G. QUIGLEY, professor of mathematics, College of Saint Teresa, Winona, Minn.

J. A. SALLADE, instructor in mathematics, Pennsylvania State College, State College, Pa.

MARION E. STORK, professor of mathematics and astronomy, Meredith College, Raleigh, N. C.

MARIAN M. TORREY, teacher of mathematics, St. Johnsbury (Vt.) Academy.

WARREN WEAVER, assistant professor of mathematics, Throop College of Technology, Pasadena, Calif.

W. H. WILSON, instructor in mathematics, Massachusetts Institute of Technology, Cambridge, Mass.

S. D. ZELDIN, instructor in mathematics, College of Hawaii, Honolulu, Hawaii.

To institutional membership:

ILLINOIS WESLEYAN UNIVERSITY, Bloomington, Ill.

COLLEGE OF SAINT TERESA, Winona, Minn.

MILLSAPS COLLEGE, Jackson, Miss.

(2) Since the enlistment of a number of our members in military service may make it impossible for them to continue as active members, the Council voted as follows:

> On request, the name of any member of the Association who is in the army or navy, or is in other war service over-seas, will be retained on the roll of membership without the payment of dues. The MONTHLY will be forwarded, however, only if a definite mailing address is furnished by the member.

(3) It was voted to hold the next annual meeting in New York City in connection with the meeting of the American Mathematical Society. The mathematics faculty of Columbia University has kindly indicated its hearty approval of this plan.

(4) The Council approved the plan already announced in the December MONTHLY for a shortened register of members, containing the full list of members, including the names, academic record and addresses of those elected since the publication of the charter membership, and any corrections in names, titles or addresses that are reported to the secretary-treasurer promptly.

(5) Matters of regular business routine were referred with power to the Council's Committee on Finance, a standing committee which, in conformity with the amended constitution, is now composed of the President, Secretary-Treasurer and Manager.

(6) The Council voted to direct its future Nominating Committees, in choosing the second candidate for each office, to take into account not only the votes for each person for that particular office, but also the total vote for that person

for all the offices. (See "Comments on the Preliminary Ballot," page 67 of this issue.)

(7) Professor MARY EMILY SINCLAIR of Oberlin College was appointed Librarian of the Association.

(8) In accordance with the amendments adopted at the annual business meeting, the Council made the following appointments:

Secretary-Treasurer: W. D. CAIRNS.
Committee on Publications:
W. H. BUSSEY, Review Editor.
H. E. SLAUGHT, Manager.
R. D. CARMICHAEL, Editor-in-Chief.
Associate Editors:

R. C. ARCHIBALD,	HELEN A. MERRILL,
E. L. DODD,	U. G. MITCHELL,
OTTO DUNKEL,	R. E. MORITZ,
B. F. FINKEL,	D. A. ROTHROCK,
TOMLINSON FORT,	D. E. SMITH,
H. R. KINGSTON,	E. B. STOUFFER.

ANNUAL BUSINESS MEETING.

The secretary-treasurer announced the names of those just elected by the Council to membership. He also reported the death of the following ten members not previously reported, all charter members of the Association:

J. A. Colson, Searsport, Me.
Mrs. Elizabeth B. Davis, Nautical Almanac Office, Washington, D. C.
G. W. Hartwell, professor of mathematics, Hamline University.
C. S. Jackson, mathematical master, Royal Military Academy, Woolwich, Eng.
C. T. Levy, formerly teaching fellow, University of California.
J. W. Nicholson, professor of mathematics, Louisiana State University.
S. F. Norris, professor of mathematics, Baltimore City College.
E. W. Ponzer, assistant professor of mathematics, Stanford University.
L. E. Pratt, Tecumseh, Neb.
W. C. Wright, consulting actuary and accountant, Medford, Mass.

It was voted that the Committee on Libraries be continued and be asked to publish a list of current texts in collegiate mathematics, giving briefly the scope and character of each.

The amendments recommended by the Council (see November MONTHLY, pp. 445, 446) were adopted unanimously. It will be remembered that the main purpose of these was to divide the burdens of editorial work and to change the manner of choosing the secretary-treasurer so that this officer shall be selected by the Council rather than elected by vote of the Association.

The election of officers for the year 1918 was conducted both by mail and in person at this meeting, as provided by the constitution. The tellers (E. L. Dodd, O. J. Ramler, and H. B. Phillips, chairman) appointed by President Cajori reported the result of the balloting as follows, a total of 319 ballots having been cast, some of which were blank in whole or in part:

For President: E. V. Huntington, 215 votes; Oswald Veblen, 97 votes; scattering, seven votes.

For Vice-President: J. W. Young, 183 votes; D. N. Lehmer, 165 votes; R. C. Archibald, 154 votes; M. B. Porter, 101 votes.

For Secretary-Treasurer, W. D. Cairns, 294 votes.

For additional members of the Council to serve until January, 1921: Florian Cajori, 253 votes; G. A. Miller, 196 votes; E. J. Wilczynski, 194 votes; Elizabeth B. Cowley, 163 votes; H. E. Hawkes, 155 votes; W. H. Bussey, 130 votes; L. W. Dowling, 85 votes; A. M. Kenyon, 81 votes.

The following were accordingly declared elected:

President, E. V. HUNTINGTON, Harvard University.
Vice-Presidents, D. N. LEHMER, University of California, and
J. W. YOUNG, Dartmouth College.
Secretary-Treasurer, W. D. CAIRNS, Oberlin College.

Additional members of the Council to serve until January, 1921:

FLORIAN CAJORI, Colorado College,
ELIZABETH B. COWLEY, Vassar College,
G. A. MILLER, University of Illinois,
E. J. WILCZYNSKI, University of Chicago.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 1, 1917. The report of the auditing committee (Mary E. Sinclair, H. E. Slaught, and H. J. Ettlinger, chairman) was then made, and both reports were accepted and approved. The financial report is printed in full below.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 1, 1917.

RECEIPTS.			EXPENDITURES.		
Balance Dec. 21, 1916		$2,953.24	Publisher's bills		$1,942.37
			Charter membership list		391.47
1915 subscriptions	$ 2.00		Managing editor's office		250.87
1916 subscriptions	22.25		Other editors' postage		7.00
1916 indiv. dues	22.50		Committee on Membership		4.85
1916 instit. dues	5.40		Comm. on Math. Requirements		23.19
1917 subscriptions	399.75		Secretary-Treasurer's office:		
1917 indiv. dues	2,080.98		Postage	$137.72	
1917 instit. dues	263.00		Bond	5.00	
1917 init. fees	140.00		Office supplies	25.36	
Sale copies of MONTHLY	35.18		Express, telegrams, freight, etc.	18.93	
Sale reprints	14.43		Library expenses	33.33	
Advertising	478.92				

Exchange	.42		Clerical work	187.25	
Interest State Savgs. Bk.	77.13		Printing	101.04	
Interest Peoples Bk.	30.04		New York meeting	89.10	
			Cleveland meeting	21.10	
Total 1917 receipts		3,572.00	Initiation fees paid to sections	4.00	
					622.83
Total receipts up to 1918 business		$6,525.24	ANNALS subvention		150.00
Total expenditures		3,392.58	Total expenditures		$3,392.58
Balance on 1917 business		$3,132.66	Cash on hand, not deposited		24.00
			Checking account		380.91
Rec'd on 1918–1920 business		352.81	State Savgs. Bk. Co. account		2,337.16
			Peoples Bkg. Co. account		643.40
			Liberty Bond		100.00
Balance Dec. 1, 1917		$3,485.47	Bank balance Dec. 1, 1917		$3,485.47

Approved by the auditing committee,
H. J. ETTLINGER, *Chairman*,
MARY EMILY SINCLAIR,
H. E. SLAUGHT.

December 28, 1917.

When the accounts were closed on December 1, 1917, as was necessary in order to furnish the auditing committee a complete record, there remained on the total business for the calendar year 1917 the following items:

BILLS RECEIVABLE.		BILLS PAYABLE (either paid in December or estimated).	
Advertising	$169.00	Publisher's bills, June–Dec.	$1,560.00
1917 dues unpaid	75.00	Dec. ANNALS subvention	75.00
Back subscriptions	6.00	Init. fees due to sections	58.00
	$250.00	President's office	5.00
		Editors' postage	20.00
		Managing editor's office	50.00
		Secretary-treasurer's office	125.00
		Printing Chicago program, ballots, and application blanks	50.00
		Committee on Membership	5.00
		Comm. on Math. Requirements	100.00
		Additional postage	45.00
		Total, approximate	$2,100.00

If to the balance on 1917 business shown in this report, $3,132.66, there be added the amount of bills receivable, $250.00, and there be subtracted the estimated amount of bills payable, $2,100.00, there results an estimated final balance on 1917 business of approximately $1,280, as compared with a similarly estimated final balance a year ago of approximately $1,180. The major portion of this $1,280 is the thousand dollars which was turned over to the Association by the management of the MONTHLY when the Association was organized and which we feel must be kept intact as a reserve fund. The management is thus able to

present a gain in the year's finances of about one hundred dollars, this gain having been brought about through the increased amount of advertising secured for the MONTHLY, through the continued expenditure of much time by a goodly number of officers at no expense to the Association, and in spite of various unavoidable expenditures, particularly the cost of paper in connection with the publication of the MONTHLY, this increase amounting to about $400.

The financial management feels that this is an auspicious showing, when it is considered that we have provided for the most pressing need of the National Committee on Mathematical Requirements and have put into operation the plan of subvention of the *Annals of Mathematics* as approved by the Association, an arrangement which it is certain will redound to the mutual advantage of the *Annals* and the members of the Association. The Association seems secure on the financial side, but if we are to continue to make a positive and strong contribution to the cause of mathematics in America, the Association will need the full support of its membership and the Council will not be free from the necessity of scanning with care all expenditures from the treasury.

Five hundred dollars of the reserve fund is being invested in a Liberty Bond, the final payment being made early in January.

W. D. CAIRNS, *Secretary-Treasurer.*

COMMENTS ON THE PRELIMINARY BALLOT.

By the SECRETARY.

The Constitution and By-Laws provide that "all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year" and that for the final ballot "the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order." The spirit of the Constitution would thus seem to be that members, merely with the knowledge that a second candidate has been chosen in this manner, are expected to exercise their right of choice on the final ballot. The discretion here given to the Council enables its nominating committees to take cognizance of such considerations as uniform geographical distribution, a proper balance as between universities and colleges, large and small institutions, etc.

Three unfavorable symptoms have appeared in the working out of this system: (1) The number voting in the preliminary ballot is disappointingly small when compared with the membership of the Association. The full success of our democratic system is quite dependent on a large participation on the part of the membership. To the recurring expression of lack of knowledge of suitable candidates for nomination the obvious answer is the clear obligation resting upon each member of acquainting himself with his fellow members both within and outside his own particular region and of making representations at stated times of persons qualified for the manifold kinds of service in the Association. (2) It would seem that there is probably too concentrated a list of names voted for; the officers of the Association have continually tried, and have succeeded reason-

ably well in their endeavor, to enlist an ever increasing number of members in active service, in the conviction that there are many others well qualified to assume some of the responsibilities of the organization besides those who have as yet been chosen. Just how to accomplish this in the preliminary ballot without causing too great a scattering of votes and on the other hand without interfering with the members' free choice, is a puzzling question. A suggestion has been made that a list of names might be proposed for the various offices when the blanks for the preliminary ballot are sent to the members, this list to be large enough and representative enough so that the members would still have a real choice, yet might avoid the disadvantage mentioned under the next head, while there would also be the possibility of proposing other names than those on the suggested list. (3) The same names are often proposed for different offices, and the popular vote for a certain person is thus weaker than if concentrated upon one office. The Council has directed its nominating committees, in choosing the second candidate for each office, to take into account the total vote which each person has received for all the offices. This will materially aid in interpreting the wishes of the members; yet it would be a further improvement if this duplication of voting could be avoided.

It is to be understood that full secrecy is observed in all matters connected with the balloting. To make sure that the necessity of signing the ballots will not act as a deterrent in voting, the secretary will hereafter ask each member to return the *unsigned* ballot, putting his name on the outside of the envelope merely for the purpose of enabling the name of the member to be checked off from the membership list. Any suggestions as to improvements in the machinery of balloting will be gladly received by the secretary. The ends desired from the method adopted by the Association will be attained only if all members devote a stamp and a few moments' time, if no more, to each of the two ballots of the year.

THE SECOND ANNUAL MEETING OF THE MISSOURI SECTION.

The second annual meeting of the Missouri Section of the Mathematical Association of America was held in Kansas City on Saturday, November 17, 1917, in conjunction with the meeting of the Missouri State Teachers' Association. The session was held in the Public Library. The chairman of the Section, Professor L. D. Ames of the University of Missouri, presided.

Twenty-one visitors were present and the following fifteen members:

L. D. Ames, University of Missouri, Columbia.
Austin C. Andrews, Manual Training High School, Kansas City.
Otto Dunkel, Washington University, St. Louis.
Zoe Ferguson, Central High School and Junior College, St. Joseph.
R. R. Fleet, William Jewell College, Liberty.
B. F. Finkel, Drury College, Springfield.
Mary E. Helwig, High School, Kansas City, Kansas.

C. E. Horne, Park College, Parkville.
Thomas W. Jackson, High School, Fulton.
William A. Luby, Polytechnic Institute, Kansas City.
Paul R. Rider, Washington University, St. Louis.
William H. Roever, Washington University, St. Louis.
R. A. Wells, Park College, Parkville.
Ella Woodyard, High School, Kansas City, Kansas.
William H. Zeigel, Missouri State Normal School, Kirksville.

The officers elected for the ensuing year are Professor William H. Roever, Washington University, St. Louis, Chairman; Professor O. D. Kellogg, University of Missouri, Columbia, Vice-Chairman; Dr. Paul R. Rider, Washington University, St. Louis, Secretary-Treasurer.

The program of the meeting, with abstracts of papers, follows below. In the absence of Professors Dean and Scarborough their papers were read by title.

Abstracts of Papers.

1. *Some Properties of Plane and Spherical Triangles and their Frequent Analogies.* By Professor William H. Zeigel, Missouri State Normal School, Kirksville.

Professor Zeigel called attention to the fact that trigonometric functions and hyperbolic functions are special cases of elliptic functions, also that plane trigonometry is a special case of spherical trigonometry where the radius of the sphere becomes infinite. Formulæ of spherical trigonometry go over into corresponding formulæ of plane trigonometry. It is an interesting exercise to obtain these analogous forms by a direct method.

If T denotes the area of a plane triangle, r_1 the radius of a sphere, and $\sigma = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}$, it follows that

$$\lim_{r_1=\infty} (\sigma r_1^2) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-c)} = T,$$

where the sides are expressed in circular measure. But r_1 usually equals unity, hence it appears that σ is a form analogous to that expressing the area of a plane triangle.

For relations involving the area, sides, angles, altitudes, radii of inscribed, circumscribed, and escribed circles in the plane triangle, many analogous forms are found involving corresponding parts in the spherical triangle. Four rules pertaining to identical and co-functional relations were obtained which shorten materially the theoretical work of spherical trigonometry.

Numerous frequently analogous forms were deduced.

2. *The Value of Mathematics in Secondary Education.* By Dr. John W. Withers, Superintendent of Instruction, St. Louis.

The discussion on the foregoing paper was led by Miss Zoe Ferguson, Central High School and Junior College, St. Joseph, and Professor B. F. Finkel, Drury College, Springfield.

3. *Sundials and Skylights.* By Professor WILLIAM H. ROEVER, Washington University, St. Louis.

Professor Roever's paper began with a general discussion of sundial construction. He then showed how the methods of dialing could be used to determine the form of the skylight in the conical roof of a circular prison so that every cell would receive the direct rays of the sun either from exterior windows or through the skylight, for a portion of the day the year around.

4. *Pure and Applied Mathematics in the Nineteenth Century.* By Professor G. R. DEAN, Missouri School of Mines, Rolla.

5. *The Equal Parallax Curve for Frontal and Lateral Vision.* By Dr. PAUL R. RIDER, Washington University, St. Louis.

It seems evident that those creatures having side vision, such as birds, have an advantage over those having frontal vision, such as man, in their ability to gauge the relative distances of their surroundings, since the axis of their vision is perpendicular to the direction of their motion, and hence as they move forward the apparent displacement of objects is a maximum. The equal parallax curve is a curve showing the distances that a man and a bird must move forward to give the same apparent displacement of objects against the horizon. This paper derives parametric equations of the curve. It appeared in *Science*, new series, Vol. 46, No. 1183, pp. 213–214.

6. *A Simple Derivation of the Derivatives of the Trigonometric Functions.* By Professor OTTO DUNKEL, Washington University, St. Louis.

The formulæ for the derivatives of x and y with respect to the length of arc of a curve $dx/ds = \cos\tau$, $dy/ds = \sin\tau$ can easily be developed in the early part of the calculus and this is done in several texts. These formulæ depend upon the same fundamental limit that is used in the usual development of the derivatives of the trigonometric functions and yield these derivatives when applied to the circle of unit radius. For this special curve we have $x = \cos\theta$, $y = \sin\theta$, $s = \theta$, and $\tau = \pi/2 + \theta$, as is easily seen from a figure. On substituting these values in the formulæ above, we have at once the derivative of $\cos\theta$, and of $\sin\theta$. It is readily seen that this deduction holds for any angle θ.

7. *The Graphical Solution of a Cubic Equation having Complex Roots.* By Mr. WILLIAM A. LUBY, Polytechnic Institute, Kansas City.

Mr. Luby's paper assumed the cubic in the form $y \equiv x^3 + px + q = 0$. The three values of x which make y zero are of the form $a \pm ib$ and c. Then it follows that

$$a + ib + a - ib - c = 0, \tag{1}$$

$$(a + ib)(a - ib) + (a + ib)c + (a - ib)c = p, \tag{2}$$

$$(a + ib)(a - ib)c = q - y. \tag{3}$$

From (1), (2), and (3), we have

$$y = -8a^3 - 2ap + q, \tag{4}$$

$$y = q \pm \frac{2}{3}\sqrt{\frac{b^2 - p}{3}}(4b^2 - p). \tag{5}$$

In (4) a and y are variables, in (5) b and y. Constructing the graphs of (4) and (5) we obtain readily the complex roots of $x^3 + px + q = 0$.

8. *Applied Mathematics for the Average Student.* By Professor J. H. SCARBOROUGH, Missouri State Normal School, Warrensburg.

9. *The Solution of Linear Differential Equations with Periodic Coefficients.* By Dr. JAMES E. MCATEE, William Jewell College, Liberty.

Dr. McAtee made a report on a paper by Professors F. R. Moulton and W. D. MacMillan of the University of Chicago on "The Solution of Certain Types of Differential Equations with Periodic Coefficients." This paper is to be found in Vol. 33 of the *American Journal of Mathematics*.

PAUL R. RIDER,
Secretary.

REPORT OF ORGANIZATION OF THE ILLINOIS SECTION.

At the call of Professor H. E. Slaught the following persons met in Room 38 of Ryerson Physical Laboratory, University of Chicago, at 4:30 P.M., on December 27, 1917, to discuss a plan for organizing an Illinois Section of the Mathematical Association of America: I. A. Barnett, G. A. Bliss, H. E. Cobb, C. E. Comstock, M. W. Coultrap, A. R. Crathorne, D. R. Curtiss, A. Emch, J. A. Foberg, J. O. Hassler, Miss Nelle Ingels, E. B. Lytle, M. McNeill, H. L. Olson, H. L. Rietz, Mr. Scheibler, J. B. Shaw, C. H. Sisam, E. J. Townsend, and E. J. Wilczynski.

Mr. J. A. Foberg, of Crane Junior College, Chicago, was unanimously elected chairman of the meeting; he appointed Mr. E. B. Lytle secretary.[1] A motion of Professor Townsend that an Illinois Section be organized was unanimously adopted. Among other plans it was suggested but not determined that the meetings of this new Illinois Section be held at the time and place of the meetings of the Illinois Academy of Science.

On motion of Professor Wilczynski the chairman was authorized to appoint four members who with the chairman shall constitute an executive committee with full power to arrange the time, place and program for the first meeting of the Illinois Section. In addition to the chairman, the members of this committee are: Nelle L. Ingels, Greenville College; C. E. Comstock, Bradley Polytechnic Institute; G. T. Sellew, Knox College; and L. S. Shively, Mount Morris College.

[1] Mr. Lytle was appointed secretary at the close of this meeting, so that these minutes are written up from memory.

On motion of Professor Bliss the executive committee just formed was directed to appoint a committee of three to report at the first meeting and to present nominations for permanent officers of the Section.

ERNEST B. LYTLE,
Secretary.

BOOK REVIEW.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

A First Course in Higher Algebra. By HELEN A. MERRILL and CLARA E. SMITH. The Macmillan Company, New York, 1917. xiv + 247 pages. $1.50.

In this work written with the idea that "Higher Algebra, to be worthy of the name, must employ advanced methods," we find the plan well carried out. It is written, too, with the conscious "hope that there is nothing to be unlearned in later work." Not only have the authors succeeded in expressing the material in form which will not have to be unlearned, but this text gives us an illustration of the fact that methods often reserved for advanced work can very profitably be applied to elementary college subjects, to the great advantage of the student for the work in hand as well as for the power gained for more advanced work, by the familiarity, secured here, with methods which will be used on more intricate material in later courses.

While this text omits very little that is given in the usual college algebra, it places before the student, in minimum space and with extreme clearness, most of the usual college algebra material and much of other material often found incidentally by the student in more advanced courses, or not found at all. As an instance of the careful presentation of a subject often slighted, and sometimes omitted, could be mentioned the chapters on the number system. The first of these, the chapter on rational numbers, precedes other work, and contains, in addition to the usual material, some of the simple theorems on integers, which are very useful for factoring; while the chapter on irrational numbers comes later where it can receive a careful treatment by the theory of limits, sufficiently detailed to secure the student a definite and clear understanding.

In the second chapter, which does not differ much from the usual somewhat condensed work on permutations, combinations, and probability, the student begins work on problems of which there is a wholesome number throughout the text. Many of the problems given in various chapters serve for the further development of the text.

While the chapter on determinants is clear and well developed on the whole, articles 48 and 50 suggest conflicting ideas of a minor. It seems to the reviewer somewhat unfortunate that the student is not given a greater familiarity with determinants of order higher than the third, since determinants have so many interesting properties that might well serve as one of the subjects "to arouse curiosity and to lead students to find for themselves what lies just beyond."

It is at this point in the text that the discussion of the theory of limits begins, with a chapter on variables and their limits, followed by a chapter on differentiation of algebraic functions. A clear geometric interpretation of both the first and the second derivative is given, showing the analytical significance of the maximum, the minimum, and the point of inflection. Since the student is here given the condition for maximum and minimum, the condition might well have been used later, on page 210, to determine whether the ordinate of the minimum point of $x^3 - 7x + 7$ is positive or negative, as a preliminary to the discussion of this function by Sturm's theorem.

There follows a chapter on series where the material is advisedly chosen, well presented, and made definite by examples. This chapter gives the student familiarity with the Cauchy test as well as other tests. On this foundation rests the development of functions in series, where undetermined coefficients are treated, and Maclaurin's expansion is used with careful study of the region of convergence. The binomial theorem is shown as a special case. The multinomial theorem is not given. In the treatment of partial fractions a sufficient number of examples is given to secure the desirable dexterity to the prospective student of calculus.

With the theory of limits, series, and convergence in hand, the student is now prepared for the chapter on irrational numbers, wherein the ideas of a dense set and a continuous set are made clear. The existence of non-terminating, non-repeating decimal numbers is shown, and it is explained how a sequence of numbers is said to define an irrational number. On this basis $\sqrt{2}$ is given and a method of finding the value of π is indicated. The sum, difference, product, and quotient of irrational numbers, the power and root of an irrational number, are also defined by sequences. The meaning of an irrational exponent is explained by means of a limit, and given with sufficient detail to make the following chapter on logarithms clear and full of meaning. While e and π are given their classification in the chapter on irrational numbers, it is in the work on logarithms that e is shown to be the $\lim_{n=\infty} [1 + (1/n)]^n$, in which work the student has another opportunity to increase his familiarity with convergence tests.

The derivative of the Naperian logarithm of a function is given, allowing the student a working knowledge of that derivative, which, though all the foundational work on interchange of limits is not included, gives the student the material for securing the logarithmic series by Maclaurin's expansion. Here, again, there arises an opportunity to use the convergence tests to find the region of convergence of the series thus secured. Not only will this chapter probably fulfil the hope of the authors that the "work will help the student to appreciate the labor that has gone into the construction of tables of logarithms," but it must surely also give the student a more accurate idea of the logarithm as a function instead of merely as a mechanical convenience.

In addition to the usual work on complex numbers as to their combination and the geometrical representation of the numbers and of their combinations, the authors prove de Moivre's theorem and give the geometrical interpretation

of powers and roots of complex numbers. The graphing of a function of a complex variable is indicated, as is also the use of complex numbers and the complex variable in series, with graphical illustration of the sequence S_n of a series. Here, as at the end of most chapters, good references are given from which the student may secure further knowledge on the subject treated.

The more or less usual chapter on the theory of equations is given, with an exposition of the theory of Sturm's functions and the approximation of real roots by Horner's method.

At the end of the book there is a very brief chapter on integration as the inverse of differentiation. The reviewer would approve the suggestion of the authors that this chapter "may serve for reference in later work;" for, while the chapter is interesting, it is hardly essential or even useful to the student of algebra. It would, no doubt, be of some use to a student of science, but such student would need so much more thorough work on integration, gained only by more complete study and extended practice with that very important operation, that it might be wise to devote the time which would be required for this chapter to establishing a more thorough acquaintance with material already in hand.

The following misprints were noted:

Page *XIII*, line 15, exponent of x should be $n - 1$ instead of $m - 1$; page 116, equation (2), numerator should be $2 - 3x + 4x^3$ instead of $2 - 3x - 4x^3$; page 121, line 18, read *positive* for *position*; page 156, line 9, read $\Delta y/\Delta x$ instead of $\Delta y/\Delta$; page 234, line 29, read $a_0x^n + a_1x^{n-1}$ instead of $a_0x^n + a_1x^{n-}$.

The material of the text, somewhat out of the usual line, chosen with discrimination, is arranged in a way to render the text easily read and of easy use for reference, thus making the book a valuable addition to the library of the undergraduate for reference, as well as a workable text for a class in the early undergraduate years.

MARY E. WELLS.

VASSAR COLLEGE.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2670. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A telegraph wire, weighing one tenth pound per yard, is stretched between poles on level ground, so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

2671. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find two rectangular parallelopipedons whose edges are rational whole numbers and whose solid diagonals are also rational whole numbers and equal.

2672. Proposed by E. T. BELL, Seattle, Washington.

There is an identity in z, (1) $A(z) \equiv B(z)C(z)$; (*e. g.*, $A(z) = 1/(1 - k^2z^2)$; $B(z) = 1/(1 - kz)$; $C(z) = 1/(1 + kz)$); and the formal expansions $A(z) = \Sigma a(n)z^n$, $B(z) = \Sigma b(n)z^n$, $C(z) = \Sigma c(n)z^n$, $(n = 0, 1, \cdots, \infty)$, when substituted in (1), give, on equating coefficients, (2): $a(n) = b(n)c(0) + b(n - 1)c(1) + \cdots + b(0)c(n)$. If (2) is an identity in n, justify such

a use of non-convergent series to obtain it (*e. g.*, for $|k| \geqq 1$ in the above). This method of finding important identities (2) has been used freely by Hermite and many others without question of its validity, and without offering independent proofs of (2).

2673. Proposed by WILLIAM O. BEAL, University of Minnesota.

A plane through the center of an oblate spheroid makes an angle, i, with the plane of its equator. Express the eccentricity, e', of this section in terms of the eccentricity, e, of a meridian section and the angle, i.

2674. Proposed by J. O. MAHONEY, Forest Avenue High School, Dallas, Texas.

If two sides of a triangle differ by less than a certain length, e, the two opposite angles will differ by less than a certain quantity λ, expressed in degrees, such that $\lambda < 61e/a$, where a expresses, with a possible error e, the length of the apparently equal sides of the triangle.

2675. Proposed by E. B. ESCOTT, Kansas City, Mo.

Sum the series

$$-\frac{1}{2}\cdot\frac{1^3x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{2^3x^5}{5}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{3^3x^7}{7}+\cdots.$$

2676. Proposed by EDWIN R. SMITH, State College, Pa.

Find the maximum term of the series

$$\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(-r,\ -sq,\ sp-r+1,\ 1)$$

where s, r, sq and sp are positive integers, $r < s$, p and q are proper fractions such that $p+q=1$, and $F(-r,\ -qs,\ sp-r+1,\ 1)$ is a hypergeometric series. If s, r and $s-r$ are large, show that the maximum term is approximately equal to

$$\sqrt{\frac{s}{2\pi pqr(s-r)}}.$$

2677. Proposed by R. K. MORLEY, Worcester, Mass.

A quarter-mile track is to be constructed, having semi-circular ends and straightaway sides. It is required to have the rectangular part of enclosed field referred to in No. 12, Granville's Calculus, page 116, as large as possible. Find length of the straightaways.

Also, required to inscribe the maximum rectangle in a track of length l, with semi-circular ends and straightaway sides, assuming that two sides of the rectangle are parallel to the straightaways. Find the length of the straightaways and the dimensions of the rectangle.

2678. Proposed by C. F. GUMMER, Queen's University, Kingston, Canada.

Find necessary and sufficient conditions that the roots of the equation

$$x^{n+1}+a_1x^n+a_2x^{n-1}+\cdots+a_{n+1}=0$$

may be all real and separated by the roots of $x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+b_n=0$.

2679. Proposed by J. W. LASLEY, University of North Carolina.

Show that the perpendicular from any point on a circle to any chord of the circle is a mean proportional to the perpendiculars from that point to the tangents at the ends of the chord.

SOLUTIONS OF PROBLEMS.

Note.—Prepare solutions as follows: (1) Give the number of the problem with the name of the proposer and his address; (2) restate the problem as published; (3) then give the name of the solver with address; (4) then give the solution carefully and neatly written.

Solutions of 477 (Algebra) were received from E. B. Escott and E. H. Worthington; of 478 (Algebra) from E. B. Escott; and of 481 (Algebra) from L. C. Mathewson. These solutions were received after copy had been prepared for publication. Professor Worthington should have received credit also for solving 476 (Algebra).

484 (Algebra). Proposed by E. V. HUNTINGTON, Cambridge, Mass.

Show that

$$\frac{\frac{1}{m^2}-\frac{k_1}{(m+1)^2}+\frac{k_2}{(m+2)^2}-\cdots+\frac{(-1)^k}{(m+k)^2}}{\frac{1}{m}-\frac{k_1}{m+1}+\frac{k_2}{m+2}-\cdots+\frac{(-1)^k}{m+k}}=\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k}$$

for all positive integral values of m and k. Here

$$k_1=\frac{k}{1},\quad k_2=\frac{k(k-1)}{1\cdot 2},\quad k_3=\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3},\quad \text{etc.}$$

This equation was suggested to the proposer by a professor of chemistry who wishes to make use of the equation, if correct, in an actual problem in bacteriology.

I. Solution by David F. Barrow, Sheffield Scientific School.

The denominator may be summed as follows: Consider

$$\int_0^1 x^{m-1}(1-x)^k dx.$$

If we expand the integrand by the binomial theorem and integrate term by term, we obtain the denominator. But if we integrate by parts, taking $u=(1-x)^k$ and $dv=x^{m-1}dx$, we obtain

$$\frac{x^m(1-x)^k}{m}\Big]_0^1+\frac{k}{m}\int_0^1 x^m(1-x)^{k-1}dx.$$

The closed part vanishes at both limits. This furnishes a convenient reduction formula by successive applications of which the integration may be completed without difficulty. We are thus led to the result,

$$\frac{1}{m}-\frac{k_1}{m+1}+\frac{k_2}{m+2}-\cdots+\frac{(-1)^k}{m+k}=\frac{k!(m-1)!}{(m+k)!}.$$

In the formula to be proved, let us multiply up this denominator on the right side.

$$(1)\quad \frac{1}{m^2}-\frac{k_1}{(m+1)^2}+\frac{k_2}{(m+2)^2}-\cdots+\frac{(-1)^k}{(m+k)^2}=\frac{k!(m-1)!}{(m+k)!}\left(\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{m+k}\right).$$

Now if $k=0$ or if $k=1$, this formula is easily seen to be true for all positive integral values of m. We shall complete the proof by mathematical induction if we establish that *if* (1) *is true for all values of* m *and any particular value of* k, *then it is true for all values of* m *and that value of* k *increased by unity.*

We assume, then, that (1) is an identity in m when $k=k$. We may therefore replace m by $m+1$, obtaining

$$(2)\quad \frac{1}{(m+1)^2}-\frac{k_1}{(m+2)^2}+\frac{k_2}{(m+3)^2}-\cdots+\frac{(-1)^k}{(m+k+1)^2}=\frac{k!m!}{(m+k+1)!}\left(\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k+1}\right).$$

Now $k_1, k_2, \cdots$ are the binomial coefficients. Hence they satisfy the relation $k_{i-1}+k_i=(k+1)_i$ (preserving the notation of the proposer). Remembering this, we subtract equation (2) from equation (1), and get

$$\frac{1}{m^2} - \frac{(k+1)_1}{(m+1)^2} + \frac{(k+1)_2}{(m+2)^2} - \cdots + \frac{(-1)^{k+1}}{(m+k+1)^2}$$

$$= \frac{k!(m-1)!}{(m+k)!}\left[\frac{1}{m} + \left(1 - \frac{m}{m+k+1}\right)\left(\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+k}\right)\right.$$

$$\left. - \frac{m}{m+k+1} \cdot \frac{1}{m+k+1}\right]$$

$$= \frac{k!(m-1)!}{(m+k)!}\left[\frac{(m+k+1)^2 - m^2}{m(m+k+1)^2} + \frac{k+1}{m+k+1}\left(\frac{1}{m+1} + \cdots + \frac{1}{m+k}\right)\right]$$

$$= \frac{(k+1)!(m-1)!}{(m+k+1)!}\left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+k+1}\right).$$

But this is what we get if in (1) we replace k by $k+1$. Hence (1) is true for all values of m and for $k+1$; hence it is true for all values of m and k.

II. Solution by Edwin Bidwell Wilson, Massachusetts Institute of Technology.

The equation is an identity for all values of m and is merely a special case of a general theorem in logarithmic differentiation of rational fractions.

Suppose that

$$A\,\frac{(x-a_1)(x-a_2)\cdots(x-a_m)}{(x-b_1)(x-b_2)\cdots(x-b_n)}$$

is a rational fraction, and for simplicity let it be proper ($m < n$) and without multiple factors in either numerator or denominator (the result for the general rational fraction is more complicated only in notation).

The derivative of the logarithm of this fraction is

$$\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_m} - \frac{1}{x-b_1} - \frac{1}{x-b_2} - \cdots - \frac{1}{x-b_n}.$$

The fraction may, however, be resolved into partial fractions as

$$\frac{c_1}{x-b_1} + \frac{c_2}{x-b_2} + \cdots + \frac{c_n}{x-b_n}$$

and its logarithmic derivative then has the form

$$-\frac{\dfrac{c_1}{(x-b_1)^2} + \dfrac{c_2}{(x-b_2)^2} + \cdots + \dfrac{c_n}{(x-b_n)^2}}{\dfrac{c_1}{x-b_1} + \dfrac{c_2}{x-b_2} + \cdots + \dfrac{c_n}{x-b_n}},$$

which must be identical with the former expression.

In the particular case mentioned in the problem the rational fraction is evidently of the form

$$\frac{A}{m(m+1)(m+2)\cdots(m+k)},$$

m replacing x. The expansion into partial fractions is

$$\frac{k_0}{m} + \frac{k_1}{m+1} + \frac{k_2}{m+2} + \cdots + \frac{k_k}{m+k}$$

and the coefficients k_p may readily be found by the "substitution" method. Indeed, if we clear of fractions and set $m = -p$ we have

$$A = k_p(-p)(-p+1)\cdots(-1)(1)(2)\cdots(k-p)$$

or

$$k_p = \frac{(-1)^p A}{p!(k-p)!} = (-1)^p \frac{k(k-1)\cdots(k-p+1)}{1\cdot 2\cdots p}$$

if the as yet undetermined A be taken as $k!$.

Another interesting special case is found on treating

$$\frac{A}{(m-k)(m-k+1)\cdots(m-1)m(m+1)\cdots(m+k-1)(m+k)},$$

especially if k be allowed to become infinite (cotangent series).

Also solved by FRANK IRWIN, C. F. GUMMER, S. BEALTY, O. SCHMIEDEL, OLIVE C. HAZLETT, and RALPH KEFFER.

485 (Algebra). Proposed by J. L. WALSH, Madison, Wisconsin.

Is it true that to every convergent series of positive terms, $a_1 + a_2 + a_3 + \cdots$, there corresponds a series of the type

$$\frac{M}{1^p} + \frac{M}{2^p} + \frac{M}{3^p} + \cdots,$$

such that $M/k^p > a_k$, $p > 1$?

SOLUTION BY E. H. MOORE, The University of Chicago.

A convergent series of positive terms, $a_1 + a_2 + \cdots + a_k + \cdots$, is a positive-valued function α; $\alpha(k) = a_k$ $(k = 1, 2, 3, \cdots)$, of the variable positive integer k, satisfying the condition that the corresponding series is convergent. Denote the class of all such functions α by $\mathfrak{C}_+$.

If the question proposed is answered in the affirmative, then there exists a sequence α_n $(n = 1, 2, 3, \cdots)$ of functions α_n of the class $\mathfrak{C}_+$, viz., the sequence of functions

$$\alpha_n : \alpha_n(k) = \frac{n}{k^{1+(1/n)}} \quad (k = 1, 2, 3, \cdots),$$

of such a nature that every function α of the class $\mathfrak{C}_+$ is dominated by a suitably chosen function α_n of the sequence, viz., $|\alpha(k)| \leqq |\alpha_n(k)|$ $(k = 1, 2, 3, \cdots)$,—that is, the class $\mathfrak{C}_+$ has the dominance property D_2 defined (for the general class of real-valued functions on the general range) in § 22 of my *Introduction to a Form of General Analysis* (The New Haven Mathematical Colloquium, Yale University Press, 1910, p. 41).

Now I have proved (§ 23c (5), *loc. cit.*) that the class $\mathfrak{M}^{\mathrm{III}_1}$ (and, a fortiori, the class $\mathfrak{C}_+$) of all absolutely convergent series of real-valued terms fails to have the dominance property D_2. Hence, the question proposed must be answered in the negative.

The proposition cited is one of a number of theorems involving various dominance properties. The present question may be answered still more luminously by citing the theorem of Hadamard (Acta Mathematica, vol. 18, 1894, p. 328, theorem (β); cf. also *loc. cit.*, p. 49) that for every sequence (α_n) of functions of the class $\mathfrak{C}_+$ such that for every k $\alpha_n(k)$ increases with n there exists a function α of the class $\mathfrak{C}_+$ of such a nature that for every n

$$\lim_{k=\infty} \frac{\alpha_n(k)}{\alpha(k)} = 0.$$

Also solved by ELIJAH SWIFT.

486 (Algebra). Proposed by FLORENCE P. LEWIS, Baltimore, Md.

Find the condition which must be satisfied by the coefficients of the quartic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

in order that the equation be solvable by successive applications of the quadratic formula.

SOLUTION BY THE PROPOSER.

The equation must be of the form $A(ax^2 + bx + c)^2 + B(ax^2 + bx + c) + C = 0$ or $Ay^2 + By + C = 0$. Let x_1 and x_1' correspond to root y_1, and x_2 and x_2' to root y_2, respectively. We then have $x_1 + x_1' = -b/a$ and $x_2 + x_2' = -b/a$. This says that the line joining x_1 and x_1' is bisected at $-b/2a$, or that x_1 and x_1' are harmonic conjugates as to $-b/2a$ and ∞. The same

is true of x_2 and x_2'. Hence, ∞ is a root of the sextic covariant, whose leading coefficient must therefore vanish. This gives $8a_0^2a_3 - 4a_0a_1a_2 + a_1^3 = 0$ as the required condition. The same result may be readily obtained by elementary methods.

Also solved by HERBERT N. CARLETON, OTTO J. RAMLER, and HORACE OLSON.

E. B. ESCOTT sent in three solutions of 515 (Geometry). The first two solutions were solutions given in E. CATALAN'S *Théorèmes et Problèmes de Géométrie Élémentaire*, pp. 237–239. The third is a neat original solution.

516 (Geometry). Proposed by R. M. MATHEWS, Riverside, California.

Through the edges of a trihedral angle planes are passed orthogonal to the opposite faces Prove them coaxial.

I. SOLUTION BY NATHAN ALTSHILLER, University of Oklahoma.

The planes x, y, z which pass through the edges OA, OB, OC of the trihedral angle O–ABC and which are perpendicular to the opposite faces a, b, c, cut these faces along lines OD, OE, OF, respectively. A plane p perpendicular to OA cuts the edges OA, OB, OC in the points A, B, C, and the lines OD, OE, OF in the points D, E, F. (The reader may readily construct the figure.)

The plane x is perpendicular to a by construction, and to the plane p, because x passes through OA; hence, x is perpendicular to the line of intersection BC of p with a, and therefore BC is perpendicular to AD.

The plane b is perpendicular to y by construction and to the plane p, because b passes through OA; hence, b is perpendicular to the line of intersection BE of y with p, and, therefore, BE is perpendicular to AC. For similar reasons, CF is perpendicular to AB.

The three altitudes AD, BE, CF of the triangle ABC concur, according to a well-known proposition, in a point H, the orthocenter of ABC; hence H is a common point of the planes x, y, z. Now these three planes have obviously the point O in common, hence they pass through the line OH.

Incidentally we have also proved: *The locus of the orthocenter of the triangle determined by three concurrent lines and a variable plane perpendicular to one of the given lines is a straight line concurrent with the given lines.*

II. SOLUTION BY W. WOOLSEY JOHNSON, Annapolis, Md.

Referred to the sphere, the problem is that of the existence of the orthocenter of the spherical triangle ABC.

Let $CD = p$ be the perpendicular from C upon AB and let the perpendicular from A upon BC cut it in O.

From the right triangle AOD we have

$$\tan OD = \tan OAD \sin AD. \tag{1}$$

From AEB, we have

$$\cot OAD = \cos c \tan B. \tag{2}$$

From ADC, we have

$$\sin AD = \tan p \cot A. \tag{3}$$

Dividing (3) by (2) and substituting in (1), we have $\tan OD = \tan p \cot A \cot B \sec c$.

Interchanging A and B, c and p remain unchanged; hence, the perpendicular from B cuts off from p the same segment OD or the three perpendiculars meet in a common point.

Also solved by HORACE OLSON, L. E. LUNN, C. J. PAYNE, WILLIAM HOOVER, and the PROPOSER.

517 (Geometry). Proposed by R. P. BAKER, University of Iowa.

The coördinates of the vertices of a regular icosahedron can be expressed rationally in terms of $(\sqrt{5}-1)/4$ and $\sqrt{(5+\sqrt{5})/8}$, that is, $\cos(2\pi/5)$ and $\sin(2\pi/5)$. Prove (1) that the cosine alone is sufficient; (2) that the irrationalities cannot be reduced further. (The theorem that they cannot be rational is proved in books on crystal theory.)

SOLUTION BY C. F. GUMMER, Queen's University, Kingston.

Since the coördinates of the center are expressible rationally in terms of the vertices, we may suppose it taken as origin. Moreover the vertices are all rational in terms of any three adjacent ones A, B, C; for a fourth is the reflection of A for the plane OBC, and so on. Suppose first that A, B, C have coördinates (o, o, r), $(r\sin\alpha, o, r\cos\alpha)$, $(r\sin\alpha\cos(2\pi/5), r\sin\alpha\sin(2\pi/5), r\cos\alpha)$. Since $AB = BC$, we find $\cos\alpha = 1/\sqrt{5}$; and α is the angle subtended at O by an edge. Hence, the general equations for A, B, C (with the origin for center) are

$$x_1^2+y_1^2+z_1^2 = x_2^2+y_2^2+z_2^2 = x_3^2+y_3^2+z_3^2 = \sqrt{5}(x_1x_2+y_1y_2+z_1z_2)$$
$$= \sqrt{5}(x_2x_3+y_2y_3+z_2z_3) = \sqrt{5}(x_3x_1+y_3y_1+z_3z_1).$$

Evidently no rational solution exists, which proves the second part of the theorem.

A simple solution is found by assuming $x_1 = y_2 = z_3 = u$, $x_2 = y_3 = z_1 = v$, $x_3 = y_1 = z_2 = w$, so that the equations become $u^2+v^2+w^2 = \sqrt{5}(uv+vw+wu)$. If $w = 0$, $u^2+v^2 = \sqrt{5}\,uv$, which is satisfied by $u = 1$, $v = 2\cos(2\pi/5)$. Therefore, A, B, C may be taken to be $(1, 0, 2\cos(2\pi/5))$, $(2\cos(2\pi/5), 1, 0)$, $(0, 2\cos(2\pi/5), 1)$.

518 (Geometry). Proposed by ROGER A. JOHNSON, Cleveland, Ohio.

If one angle of a triangle is 60^0, the Euler line (the line through the circumcenter, orthocenter, and median point) is perpendicular to the bisector of that angle; and if one angle is 120^0, the Euler line is parallel to the bisector of that angle.

SOLUTION BY J. L. COOLIDGE, Harvard University.

Let the vertices of the triangle be A_1, A_2, A_3, the middle points of the sides M_1, M_2, M_3, the feet of the altitudes H_1, H_2, H_3, the circumcenter O, and the orthocenter H.

Let us take A_1 as the angle in which we are interested, and assume it first to be acute.

The angle which the external bisector forms with A_2A_3 is $\frac{1}{2}(A_2-A_3)$ and its tangent

$$\tan\tfrac{1}{2}(A_2-A_3) = \frac{1-\cos(A_2-A_3)}{\sin(A_2-A_3)}.$$

The tangent of the angle which the Euler line makes with A_2A_3 is

$$\frac{OM_1-HH_1}{A_2M_1-A_2H_1}.$$

We have

$$OM_1 = r\cos A_1 = r[\sin A_2\sin A_3-\cos A_2\cos A_3],$$
$$A_2M_1 = r\sin A_1 = r[\sin A_2\cos A_3+\cos A_2\sin A_3],$$
$$HH_1 = (A_2H_1)\operatorname{ctn} A_3 = 2r\cos A_2\cos A_3,$$

and

$$A_2H_1 = a_3\cos A_2 = 2r\cos A_2\sin A_3.$$

The tangent of this angle is

$$\frac{\sin A_2\sin A_3-3\cos A_2\cos A_3}{\sin(A_2-A_3)}.$$

The two tangents are equal if

$$1-\cos A_2\cos A_3-\sin A_2\sin A_3 = \sin A_2\sin A_3-3\cos A_1\cos A_3, \text{ or } \cos(A_2+A_3) = -\tfrac{1}{2}.$$

Hence,

$$A_2+A_3 = 120^0.$$

The case when A_1 is obtuse may be treated by a similar method.

Also solved by A. M. HARDING, NATHAN ALTSHILLER, C. C. YEN, J. F. LÜ, K. K. CHAN, HORACE OLSON, G. BREIT, H. C. GOSSARD, FRANK V. MORLEY, LOUIS WEISNER, and OTTO J. RAMLER.

428 (Calculus). Proposed by J. L. RILEY, Northwestern State Normal School, Tahlequah, Okla.

The loop of a lemniscate rolls in contact with the axis of x. Prove that the locus of the node is given by the equation

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{a}{y}\right)^{4/3},$$

and that $2\rho\rho' = a^2$, if ρ, ρ' be corresponding radii of curvature of this locus and the lemniscate.

SOLUTION BY A. M. HARDING, University of Arkansas.

The problem, as originally stated, is incorrect. It should read as above. The equation of the lemniscate, referred to a tangent at its center, is $r^2 = a^2 \sin 2\theta$. By the formulas of the elementary calculus it is easily shown that

$$s = \int_0^\theta a\sqrt{\csc 2\theta}\, d\theta, \qquad \psi = 2\theta, \qquad \text{and} \qquad \rho = \frac{a}{3}\sqrt{\csc 2\theta},$$

where s is the length of arc from the node $A(x, y)$ to the point P, ψ is the angle between AP and the tangent at P, and ρ is the radius of curvature at P.

Let the lemniscate roll along the x-axis until the tangent at P coincides with this axis, and let O be the origin of coördinates. Then

$x = OP - AP\cos\psi = a\int_0^\theta \sqrt{\csc 2\theta}\, d\theta - a\sqrt{\sin 2\theta}\cos 2\theta$, and $y = AP\sin\psi = a\sqrt{\sin 2\theta}\sin 2\theta$; whence,

$$\frac{dy}{dx} = \cot 2\theta,$$

and

$$\frac{d^2y}{dx^2} = -\frac{2(\csc 2\theta)^{4/2}}{3a\sin^2 2\theta}.$$

Hence,

$$1 + \left(\frac{dy}{dx}\right)^2 = \csc^2 2\theta = \left(\frac{a}{y}\right)^{4/3},$$

and

$$\rho' = -\frac{3}{2}a\sqrt{\sin 2\theta}.$$

Hence,

$$\rho\rho' = -\frac{a^2}{2}.$$

Also solved by WILLIAM HOOVER.

430 (Calculus). Proposed by G. PAASWELL, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z-axis and pass a plane through the x- (or y-) axis. Find the area of this surface intercepted by this plane and the xz- (or yz-) plane.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let a = the radius of the circle, c = the distance of the chord from the center, 2α = the angle subtended at the center by the arc of the chord; take the middle of the chord for the origin, and the radius at right angles to the chord for the x-axis; the equation to the generating arc is then

$$z^2 + (x + a\cos\alpha)^2 = a^2,$$

or

$$x = \sqrt{a^2 - z^2} - a\cos\alpha.$$

The element of length of the arc is $ds = adz/\sqrt{a^2 - z^2}$, and the required surface

$$S = \int 2\pi x ds = 2\pi \int_0^{a\sin\alpha} (\sqrt{a^2 - z^2} - a\cos\alpha)\frac{adz}{\sqrt{a^2 - z^2}}$$

$$= 2\pi a\left[z - a\cos\alpha \int \frac{dz}{\sqrt{a^2 - z^2}} \right]_0^{a\sin\alpha}$$

$$= 2\pi a^2\left[\sin\alpha - \cos\alpha \left(\sin^{-1}\frac{z}{a} \right)_0^{a\sin\alpha} \right] = 2\pi a^2(\sin\alpha - \alpha\cos\alpha).$$

If $\alpha = \pi/2$, $S = 2\pi a^2$, as it should be.

The object of the choice of coördinate axes as assigned in the statement of the problem is not evident.

The volume is

$$V = \pi \int x^2 dz = \pi \int_0^{a\sin\alpha} (\sqrt{a^2 - z^2} - a\cos\alpha)^2 dz$$

$$= \pi\left[a^2(1 + \cos^2\alpha)z - \tfrac{1}{3}z^3 - 2a\cos\alpha\left(\frac{z}{2}\sqrt{a^2 - z^2} + \frac{a^2}{2}\sin^{-1}\frac{z}{a} \right) \right]_0^{a\sin\alpha}$$

$$= \pi a\{\tfrac{1}{3}(2a^2 + a^2\cos^2\alpha)\sin\alpha - a\alpha\cdot a\cos\alpha\}$$

$$= \pi a\left(\frac{2a^2 + c^2}{3}\sin\alpha - a\alpha\cdot c \right), \qquad c = a\cos\alpha.$$

If $c = 0$, $V = \frac{2}{3}\pi a^3$, as it should be.

431 (Calculus). Proposed by J. W. LASLEY, University of North Carolina.

Explain Bertrand's fallacy:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{x^2 - y^2}{(x^2 + y^2)^2}dydx = \int_{y=0}^{y=1}\int_{x=0}^{x=1} \frac{x^2 - y^2}{(x^2 + y^2)^2}dxdy,$$

$$\tfrac{1}{4}\pi = -\tfrac{1}{4}\pi, \; 1 = -1.$$

SOLUTION BY HORACE OLSON, Chicago, Illinois.

There is a theorem that the order of integration of a double integral may be reversed *if the integrand is continuous in both variables within the region of integration.* The integrand in this fallacy is discontinuous at $(x = 0, y = 0)$. Therefore, the theorem does not justify the reversal of the order of integration.

341 (Mechanics). Proposed by PAUL CAPRON, U. S. Naval Academy.

A pole l feet long, with one end on the ground, touches the top of a wall a feet high and slides in a vertical plane perpendicular to the wall. Show that its instantaneous center of rotation is at the intersection of the vertical where it touches the ground with the perpendicular to its axis where it touches the wall, and that the locus of this center is a parabola having the latus rectum a.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let T be the point at the top of the wall and G the point on the ground through which the pole passes at some given instant. Let O be the point on the ground in the same vertical line with T, and let θ be the angle TGO.

The direction of motion of any point is clearly at right angles to the line joining that point to the instantaneous center of rotation. (See Ziwet, *Theoretical Mechanics*, Art. 23; Demartres, *Cours de Géométrie infinitésimale*, Art. 20.) Hence, if the direction of motion of a point be known, the instantaneous center C must lie on the normal at the point to the direction of motion.

Now the direction of motion of that point of the rod which at the instant coincides with T is clearly along the rod. Hence the instantaneous center C is on the normal to the rod at T. Again, the direction of motion of the point G on the ground is along the ground. Hence, C is in the vertical line through G.

To find the locus of C, we choose OG and OT as coördinate axes. Then,

$$x = OG = OT \cot\theta = a\cot\theta \text{ and } y = GC = TG\csc\theta = a\csc^2\theta.$$

Eliminating θ between these two equations, we readily obtain

$$x^2 = a(y - a),$$

which is the equation of a parabola with its vertex at T $(0, a)$, its axis vertical, and latus rectum equal to a. The pole being of limited length l, the locus of C is that part of the parabola for which $0 \leqslant x \leqslant \sqrt{l^2 - a^2}$.

Also solved by H. R. HOWARD, WILLIAM HOOVER, J. B. REYNOLDS, and the PROPOSER.

342 (Mechanics). Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform rod of length $2a$ is freely hinged at one end, at the other end a string of length b is attached which is fastened at its further end to a point on the surface of a homogeneous sphere of radius c. If the masses of the rod and sphere are equal, find the motion of the system when slightly disturbed from the vertical, and the cubic equation giving the corresponding small oscillations.

SOLUTION BY THE PROPOSER.

For symmetry of notation, let there be three bodies of masses m_1, m_2, m_3; of axes of symmetry whose lengths are $2a_1, 2a_2, 2a_3$ with centers of gravity G_1, G_2, G_3 distant a_1, a_2, a_3 from corresponding extremities; radii of gyration about G_1, G_2, G_3 equal to k_1, k_2, k_3; the origin of rectangular coördinates at the fixed extremity of the highest body, (x_1, y_1), (x_2, y_2), (x_3, y_3), the horizontal through the fixed point being the axis of x; and let $\varphi_1, \varphi_2, \varphi_3$ be the angles which the axes of symmetry of the bodies make with the vertical at any time t from the beginning of motion. Then,

$$x_1 = a_1\sin\varphi_1, \qquad y_1 = a_1\cos\varphi_1, \tag{1}$$

$$x_2 = 2a_1\sin\varphi_1 + a_2\sin\varphi_2, \qquad y_2 = 2a_1\cos\varphi_1 + a_2\cos\varphi_2, \tag{2}$$

$$x_3 = 2a_1\sin\varphi_1 + 2a_2\sin\varphi_2 + a_3\sin\varphi_3, \qquad y_3 = 2a_1\cos\varphi_1 + 2a_2\cos\varphi_2 + a_3\cos\varphi_3. \tag{3}$$

The kinetic potential equation is

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + k_1^2\dot{\varphi}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + k_2^2\dot{\varphi}_2^2) + \tfrac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2 + k_3^2\dot{\varphi}_3^2)$$
$$= m_1ga_1\cos\varphi_1 + m_2g(2a_1\cos\varphi_1 + a_2\cos\varphi_2)$$
$$+ m_3g(2a_1\cos\varphi_1 + 2a_2\cos\varphi_2 + a_3\cos\varphi_3) = V. \tag{4}$$

From (1), (2), (3),

$$\dot{x}_1 = a_1\cos\varphi_1\cdot\dot{\varphi}_1, \qquad \dot{y}_1 = -a\sin\varphi_1\cdot\dot{\varphi}_1, \tag{5}$$

$$\dot{x}_2 = 2a_1\cos\varphi_1\cdot\dot{\varphi}_1 + a_2\cos\varphi_2\cdot\dot{\varphi}_2, \qquad \dot{y}_2 = -2a_1\sin\varphi_1\cdot\dot{\varphi}_1 - a_2\sin\varphi_2\cdot\dot{\varphi}_2, \tag{6}$$

$$\dot{x}_3 = 2a_1\cos\varphi_1\cdot\dot{\varphi}_1 + 2a_2\cos\varphi_2\cdot\dot{\varphi}_2 + a_3\cos\varphi_3\cdot\dot{\varphi}_3,$$
$$\dot{y}_3 = -2a_1\sin\varphi_1\cdot\dot{\varphi}_1 - 2a_2\sin\varphi_2\cdot\dot{\varphi}_2 - a_3\sin\varphi_3\cdot\dot{\varphi}_3. \tag{7}$$

These in (4) give

$$T = \tfrac{1}{2}m_1(a_1^2 + k_1^2)\dot{\varphi}_1^2 + \tfrac{1}{2}m_2\{4a_1^2\dot{\varphi}_1^2 + 4a_1a_2\cos(\varphi_1 - \varphi_2)\dot{\varphi}_1\dot{\varphi}_2 + (a_2^2 + k_2^2)\dot{\varphi}_2^2\}$$
$$+ \tfrac{1}{2}m_3\{4a_1^2\dot{\varphi}_1^2 + 4a_2^2\dot{\varphi}_2^2 + (a_3^2 + k_3^2)\dot{\varphi}_3^2 + 8a_1a_2\cos(\varphi_1 - \varphi_2)\dot{\varphi}_1\dot{\varphi}_2$$
$$+ 4a_2a_3\cos(\varphi_2 - \varphi_3)\dot{\varphi}_2\dot{\varphi}_3 + 4a_1a_3\cos(\varphi_1 - \varphi_3)\dot{\varphi}_1\dot{\varphi}_3\}$$
$$= m_1ga_1\cos\varphi_1 + m_2g(2a_1\cos\varphi_1 + a_2\cos\varphi_2)$$
$$+ m_3g(2a_1\cos\varphi_1 + 2a_2\cos\varphi_2 + a_3\cos\varphi_3) = V. \tag{8}$$

Applying to (8) Lagrange's equations of typical form

$$\frac{d}{dt}\frac{dT}{d\dot{\varphi}_n} - \frac{dT}{d\varphi_n} = \frac{dV}{d\varphi_n}, \qquad n = 1, 2, 3, \tag{9}$$

the φ_1 function is

$$\{m_1(a_1^2 + k_1^2) + 4a_1^2m_2 + 4a_1^2m_3\}\ddot{\varphi}_1 + 2a_1a_2m_2\cos(\varphi_1 - \varphi_2)\ddot{\varphi}_2 + 2a_1a_2m_2\sin(\varphi_1 - \varphi_2)\dot{\varphi}_2^2$$
$$+ 4a_1a_3m_3\cos(\varphi_1 - \varphi_2)\ddot{\varphi}_2 + 4a_1a_2m_3\sin(\varphi_1 - \varphi_2)\dot{\varphi}_2^2 + 2a_1a_3m_3\cos(\varphi_1 - \varphi_3)\ddot{\varphi}_3$$
$$+ 2a_1a_3m_3\sin(\varphi_1 - \varphi_3)\dot{\varphi}_3^2 = -ga_1(m_1 + 2m_2 + 2m_3)\sin\varphi_1, \tag{10}$$

and the φ_2, φ_3 functions are

$$\{m_2(a_2^2 + k_2^2) + 4a_2^2m_3\}\ddot{\varphi}_2 + \{2a_1a_2m_2\cos(\varphi_1 - \varphi_2) + 4a_1a_2m_3\cos(\varphi_1 - \varphi_2)\}\ddot{\varphi}_1$$
$$- 2a_1a_2m_2\sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_1^2 - 4a_1a_2m_3\sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_1^2 + 2a_2a_3m_3\cos(\varphi_2 - \varphi_3)\ddot{\varphi}_3$$
$$+ 2a_2a_3m_3\sin(\varphi_2 - \varphi_3)\dot{\varphi}_3^2 = -ga_2(m_2 + 2m_3)\sin\varphi_2, \tag{11}$$

$$m_3\{(a_3^2 + k_3^2)\ddot{\varphi}_3 + 2a_2a_3\cos(\varphi_2 - \varphi_3)\ddot{\varphi}_2 - 2a_2a_3\sin(\varphi_2 - \varphi_3)\dot{\varphi}_2^2 + 2a_1a_3\cos(\varphi_1 - \varphi_3)\ddot{\varphi}_1$$
$$+ 2a_1a_3\sin(\varphi_3 - \varphi_1)\cdot\dot{\varphi}_1^2\} = -m_3ga_3\sin\varphi_3. \tag{12}$$

Let φ_1, φ_2, φ_3 be so small that one may employ the approximations $\sin\varphi_1 = \varphi_1$, etc.; $\cos\varphi_1 = 1$, etc.; $\dot{\varphi}_1^2 = 0$, etc.; $\dot{\varphi}_1\dot{\varphi}_2 = 0$, etc. Then (10), (11), (12) may be written

$$\{m_1(a_1^2 + k_1^2) + 4a_1^2(m_2 + m_3)\}\ddot{\varphi}_1 + 2a_1a_2m_2\ddot{\varphi}_2 + 4a_1a_2m_3\ddot{\varphi}_2 + 2a_1a_3m_3\cos(\varphi_1 - \varphi_3)\ddot{\varphi}_3$$
$$= -ga_1(m_1 + 2m_2 + 2m_3)\varphi_1, \tag{13}$$

$$\{2a_1a_2m_2 + 4a_1a_2m_3\}\ddot{\varphi}_1 + \{m_2(a_2^2 + k_2^2) + 4a_2^2m_3\}\ddot{\varphi}_2 + 2a_2a_3m_3\ddot{\varphi}_3 = -ga_2(m_1 + 2m_3)\varphi_2, \tag{14}$$

$$m_3\{(a_3^2 + k_3^2)\ddot{\varphi}_3 + 2a_2a_3\ddot{\varphi}_2 + 2a_1a_3\ddot{\varphi}_1\} = -ga_3m_3\varphi_3. \tag{15}$$

These are the three equations of small motion in the general form of the problem.

In the present problem, $m_2 = 0$; $k_1^2 = a_1^2/3$, $k_2 = 0$, $k_3^2 = \frac{2}{3}a_3^2$; and we may put $m_1 = m_3 = m$; also $2a_1 = 2a$, $2a_2 = b$, $2a_3 = 2c$, and (13), (14), (15) are

$$\tfrac{16}{3}a\ddot{\varphi}_1 + 3g\varphi_1 + 2b\ddot{\varphi}_2 + 2c\ddot{\varphi}_3 = 0,$$
$$2a\ddot{\varphi}_1 + b\ddot{\varphi}_2 + g\varphi_2 + c\ddot{\varphi}_3 = 0,$$
$$2a\ddot{\varphi}_1 + b\ddot{\varphi}_2 + \tfrac{4}{3}c\ddot{\varphi}_3 + g\varphi_3 = 0;$$

or, using another notation of differential equations,

$$(\tfrac{16}{3}aD^2 + 3g)\varphi_1 + 2bD^2\varphi_2 + 2cD^2\varphi_3 = 0,$$
$$2aD^2\varphi_1 + (bD^2 + g)\varphi_2 + cD^2\varphi_3 = 0,$$
$$2aD^2\varphi_1 + bD^2\varphi_2 + (\tfrac{4}{3}cD^2 + g)\varphi_3 = 0.$$

Eliminate φ_1 and φ_2; then,

$$\{8abcD^6 + g(20ab + 18bc + 52ac)D^4 + g^2(80a + 45b + 63c)D^2 - 45g^3\}\varphi_3 = 0. \tag{16}$$

Let

$$\varphi_3 = L_p\cos(\lambda t + \alpha_p),$$

and after substitution in (22) and simplifying, put $\lambda^2 = \mu$, and we have

$$8abc\mu^3 - g(20ab + 18bc + 52ac)\mu^2 + g^2(80a + 45b + 63c)\mu - 45g^3 = 0,$$

the cubic equation required.

Also solved by J. B. REYNOLDS and W. E. CEDERBERG.

259 (Number Theory). Proposed by E. E. WHITFORD, College of the City of New York.

If p is relatively prime to 10, and if any multiple of p consisting of n digits has its digits permuted cyclically, the number thus formed is also a multiple of p; the number n to be determined by the congruence $10^n \equiv 1 \pmod{p}$. For example, 481, 814, and 148 are each multiples of 37.

SOLUTION BY BENJAMIN F. YANNEY, College of Wooster, Wooster, Ohio.

Let the digits in the initial order be

$$a_n,\ a_{n-1},\ \cdots,\ a_1.$$

Then by hypothesis,

$$a_n 10^{n-1} + a_{n-1}10^{n-2} + \cdots + a_2 10 + a_1 \equiv 0 \pmod{p}.$$

Multiply each member of the congruence by 10, remembering that $10^n \equiv 1 \pmod{p}$, and place the digit a_n in units place. We thus secure one cyclic permutation. Another cyclic permutation is secured by multiplying again by 10; and so on. This completes the proof.

It will be observed that the theorem may be generalized by multiplying all the terms to the left of any specified term by $10^{\kappa n}$, in the case of any cyclic permutation, where κ is any positive integer. Thus, in the example given, 480001, 800000014, and 140000000008 are also each multiples of 37. By successive application of this method, we may obtain different types of cyclic permutations. Thus, 400000080001 is a multiple of 37. We may have other than cyclic permutations, with ciphers, by multiplying any one or more terms of the above congruence by $10^{\kappa n}$, where κ can have a different value for each term multiplied. Thus, 80401 is also a multiple of 37. It is interesting to note in this more general application that no two integers of the original number can ever collide.

Also solved by L. C. MATHEWSON, PHILIP FRANKLIN, W. R. RANSOM, FRANK IRWIN, PAUL CAPRON, HORACE OLSON, and C. C. YEN.

262 (Number Theory). Proposed by C. N. SCHMALL, New York City.

If x, y, z, are 3 integers, consecutive among the integers prime to 3, show that

$$x(x - 2y) - z(z - 2y) = \pm 3.$$

SOLUTION BY EDWARD H. VANCE, Graduate Student, Ithaca, N. Y.

Let $v - 1$ be any number divisible by 3, then any set of three integers consecutive among the numbers prime to 3 may be represented by one of the following sets:

$$v - 2,\ v,\ v + 1;\ \ v,\ v + 1,\ v + 3.$$

Substituting $v - 2, v, v + 1$ for x, y, z, respectively, in the lefthand side of the given equation we have

$$x(x - 2y) - z(z - 2y) = 3.$$

Substituting $v, v + 1, v + 3$ for x, y, z, respectively, we have

$$x(x - 2y) - z(z - 2y) = -3.$$

Also solved by PAUL CAPRON, N. P. PANDYA, LOUIS O'SHAUGHNESSY, LEWIS CLARK, E. F. CANADAY, GEORGE W. HARTWELL, J. L. RILEY, ALBERT G. RAU, HERBERT N. CARLETON, HORACE OLSON, and V. M. SPUNAR.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

I. ON MAKING MATHEMATICAL RESULTS MORE AVAILABLE FOR ENGINEERS.

By WILLIS WHITED, Harrisburg, Pennsylvania.

Some time ago I received a circular from the Mathematical Association of America regarding the *Annals of Mathematics*. I like very much the idea of a

series of articles setting forth the "state of the art" of the different branches of mathematics in a form that would be intelligible to people who are not specialists in the respective branches.

I am an engineer and know that there are numerous unsolved problems in engineering science which are chiefly mathematical. The engineer studies mathematics primarily for its value as a tool in solving his problems, however fond he may be of the subject for its own sake. Very few engineers find time, in the course of an ordinary lifetime, to acquire a reasonably complete knowledge of all the pure mathematics that they can use to advantage in following up the latest advances in their respective specialties and in doing the research work that devolves upon them. It not infrequently happens that work which appears at the time to be little more than mathematical gymnastics is subsequently developed into something quite useful; but years elapse before the people who need the mathematics learn of its existence. The investigating engineer and the mathematician must keep in closer touch with each other in the future than they have in the past. America must take a larger place in the advancement of science.

The engineering investigator who encounters difficult mathematical problems must have better facilities for acquiring the knowledge he needs of the many powerful methods of mathematical analysis which have been developed within the memory of men now living. Works on advanced mathematics are practically all intended for professional mathematicians. Their contents are almost wholly academic in character and they are beyond the reach of the engineer. Articles in mathematical periodicals are seldom intelligible to any but a very few specialists. This is doubtless unavoidable and perfectly proper, but I would urge that occasional articles be written bringing various branches of the subject down to date, omitting, perhaps, much of the purely academic work and expressing the whole, if possible, in terms that can be understood by the engineer who has kept up his collegiate mathematics.

From what little I know of modern mathematics, I would imagine that progress useful to the engineer has been or soon may be attained in the following branches (among others): differential equations, calculus of finite differenees, vector analysis, successive integration, elliptic and hyperelliptic functions, transcendental equations and analytical geometry.

Most of the modern writers on advanced analytical geometry use homogeneous coördinates. This method has some advantages in certain kinds of work, but it is rarely taught to undergraduates in engineering and, moreover, most of the engineer's problems are metrical, so that Cartesian coördinates are better adapted to their solution. Many theorems in projective geometry could be used by the engineer who employs graphical solutions if the theorems were put in such form that he could acquire a knowledge of them in a reasonable time.

Most of the fundamental principles of those branches of science which aspire to become exact can best be expressed in the form of differential equations. Many of these equations have not, thus far, been solved. Approximate solutions are better than none. Hence, I would urge that methods of approximate solu-

tions be so developed as to make them, so far as practicable, accessible to the engineer. In the practical applications of mathematics to engineering and, probably, to other sciences, the solutions of problems are often not exact. Graphical solutions are subject to a very considerable margin of error and arithmetical solutions almost always involve the multiplication or division of decimals in which only a certain number of decimal places are retained. Transcendental functions and radicals are only given approximately in the tables and it may well happen that a solution in a rapidly converging series is just as convenient as an exact solution. If a solution is in the form of a series with general expressions for coefficients, it may be almost as satisfactory as any other kind of a formula. In that case, if a similar problem occurs again, it will only be necessary to substitute the proper values for the constant terms in the coefficients, which can be done by an assistant who is not familiar with differential equations. I therefore hope that mathematicians will publish freely their methods for approximate solutions of differential equations and other problems, preferably in a form that will not compel the busy engineer to search through a multitude of monographs, many of which are in foreign languages and some of which can not be readily obtained, before he can get an adequate idea of the nature of the solution.

Elliptic integrals are met with occasionally and if they merely have to be integrated once approximate methods are available. If successive integration is required, it is apt to be "another story."

It may be that all problems that can be solved by vector analysis can also be solved by the older methods, but this method is often so much simpler that the subject is worthy thorough study.

The engineer often meets with transcendental equations and they usually have to be solved as individual problems. If more general methods, even if only approximate, have been developed, they should be more generally known.

Complex variables are occasionally encountered, chiefly in connection with differential equations. If a practical knowledge of the subject could be imparted without requiring the reader to toil through ponderous tomes in an effort to find an explanation, it would be helpful.

The modern theory of functions is a subject which is very interesting to one who is fond of mathematics for its own sake; but can not some way be found by which the student can get at the pith of the matter in a reasonable time? The subject is chiefly academic, but is very attractive.

II. Relating to New Remainder Terms for Certain Integration Formulæ.

By S. A. Corey, Albia, Iowa.

In the June, 1917, number of the Monthly Professor Daniell notes the fact that at least one of the remainder terms of the integration formulæ which I gave in the June–July, 1912, number of the Monthly is needlessly large. I also observe that the remainder term to my formula 25s which he gives is too small, as he has tacitly made the unwarranted assumption that the signs of his S_1 and S_2

must always be alike. To set these matters aright I have computed a new set of remainder terms (based on a form due to Poisson[1]) for all the formulæ originally given. All the work of computation has been carefully checked to eliminate errors as far as possible. We employ the symbol r_{2n},

$$r_{2n} = \pm\,\theta\,\frac{x^{2n+1}}{(2n)!}\max\left|f^{(2n+1)}(a+\theta_1 x)\right|,$$

where $0 \leq \theta \leq 1$, and $0 \leq \theta_1 \leq 1$.

Formula	Remainder Term	Formula	Remainder Term
(1)	$r_4/72$	(2)	$r_4/216$
(3)	$17r_6/24192$	(4)	$11r_6/7560$
(5)	$.000,154r_8$	(6)	$.000,000,586r_8$
(7)	$.000,000,279r_8$	(8)	$.000,057,6r_6$
(9)	$.000,009,56r_8$	(10)	$.000,011,5r_8$
(11)	$.000,029r_8$	(12)	$.000,060,6r_8$
(13)	$.000,010,7r_{10}$	(14)	$.000,015,9r_8$
(15)	$.000,000,584r_{12}$	(16)	$.000,000,252r_{12}$
(17)	$.000,000,16r_{10}$	(18)	$.000,105r_6$
(19)	$.000,29r_6$	(20)	$.000,011r_6$
(21)	$.000,000,143r_{10}$	(22)	$.000,000,000,203r_{16}$
(23)	$.000,010,6r_6$	(24)	$.000,000,002,16r_{12}$
(25)	$r_6/504$	(26)	$r_6/3024$
(27)	$.000,046,9r_8$	(28)	$.000,146r_8$
(29)	$.000,011,7r_{10}$	(30)	$.000,000,006,9r_{10}$
(31)	$.000,001,65r_8$	(32)	$.000,000,335r_{10}$
(33)	$.000,000,419r_{10}$	(34)	$.000,001,16r_{10}$
(35)	$.000,002,76r_{10}$	(36)	$.000,000,537r_{12}$
(37)	$.000,000,425r_{10}$	(38)	$.000,000,020,8r_{14}$
(39)	$.000,000,005,53r_{14}$	(40)	$.000,000,002,56r_{12}$
(41)	$.000,000,118r_8$	(42)	$.000,662r_8$
(43)	$.000,050,9r_8$	(44)	$.000,006,27r_{10}$
(45)	$.000,031,5r_{10}$	(46)	$.000,001,87r_{12}$
(47)	$.000,000,000,15r_{12}$	(48)	$.000,000,097r_{10}$
(49)	$.000,000,055r_{14}$	(50)	$.000,000,24r_{12}$
(51)	$.000,000,021r_{12}$	(52)	$.000,000,000,213r_{16}$
(53)	$.000,372r_{10}$	(54)	$.000,013r_{10}$
(55)	$.000,001,29r_{12}$	(56)	$.000,000,029,8r_{14}$.

[1] See Ford, *Studies on Divergent Series and Summability*, p. 6.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF GREENVILLE COLLEGE, Greenville, Ill.

This club was organized in September, 1916, "for the purpose of arousing interest in mathematics and of taking up subjects not usually discussed in class work." There are 25 members this year and of these about 20 usually attend each meeting. At the open meeting, addressed by Professor Rietz of the University of Illinois, between 50 and 60 persons were present. The programs of the meetings planned for 1916–17 and 1917–18 were printed and the issue of the *Greenville College Bulletin* for November 1, 1917, reproduced the latter program and a group photograph of the club members of last year.

The club levies no regular dues but assesses its members for such necessary expenses as bringing speakers from out of town, printing, making cuts for the college annual, etc.

Officers: President, James Caroway '19; vice-president, James Dilbeck '18; secretary, Edna Lindh '20; treasurer, Gerald Bergman '19; 1st consul, Herschel Lintherland '18; 2d consul, Artemisia Watkins '19. The head of the department of mathematics, Professor Nelle Louise Ingels, and the consuls, constitute the program committee.

The programs for 1916–18 are as follows:

November 12, 1916: "The Correlation in Efficiency of Mathematics and other Branches in College" by Beryl Eales '17; "Determination of the Distance of the Earth to the Sun" by Harry Schwartz '20.

December 13: "Vocational Value of Mathematical Training" by William Treloar '18; "Lives of Philosophical Mathematicians" by Richard Donoghue '18; "Zeno's Arguments" by James Dilbeck '18.

January 10, 1917: "History and Development of Logarithms" by Artemisia Watkins '19; "Slide Rule" by Ira King '18.

February 17: Address by Edgar J. Townsend, Professor of Mathematics at the University of Illinois.

March 14: "Fourth Dimension" by Gerald Bergman '19; "Mathematics of Chemistry" by Wendell Griffith '17.

April 11: "Trisection of the Angle" by George Morland; "Squaring the Circle" by Ruth Hoffmann '18; "Duplication of the Cube" by Floyd Chase '17.

November 28, 1917: "The Function of Mathematics in Scientific Research" by James Caroway '19; "How to Study" by Edna Lindh '20.

December 8: "The Mathematical Treatment of the Data of Science" by Henry L. Rietz, professor of mathematical statistics at the University of Illinois.

January 9, 1918: "Non-Euclidean Geometry" by Gerald Bergman '19; "Hyperspace" by Harry Schwartz '20.

February 6: "Medieval Courses of Mathematics" by Herschel Lintherland '18; "Correlation of Efficiency in Mathematics and Efficiency in Other High School Branches" by Professor Ingels.

March 6: "Famous Problems of Antiquity" by Artemisia Watkins '19; "Value of Calculus" by Ruth Cochran '20.

April 2: "History and Significance of the Theory of Limits in Mathematical Study" by James Dilbeck '18.

THE WHITE MATHEMATICS CLUB AT THE UNIVERSITY OF KENTUCKY, Lexington, Ky.

Although this club was founded in the autumn of 1908 "it was not given its name till four years ago.[1] The club is composed of instructors, graduate students and students majoring in the department of mathematics. Its purpose is to create and foster a sympathetic and harmonious relationship among those working in the field of mathematics, to have its members review and present in brief papers the works of the leading mathematicians, to furnish a means for the systematic and detailed study of some standard work, and, in general, to stimulate the mathematical interests of its members." Meetings are held weekly. Number of members, 7.

Officers 1917–18: President, Professor P. P. Boyd; secretary-treasurer, Professor H. H. Downing.

The program is as follows:

October 31 and November 7: "War Mathematics" by Homer L. Reid Gr.

November 14: "An Account of Scientific Measurements of the Results of Instruction" by Professor Boyd.

November 21: "Fermat's Last Theorem and the Origin and Nature of Algebraic Numbers" [review of L. E. Dickson's article in *Annals of Mathematics*, 1917] by Professor E. L. Rees.

November 28: "A Problem suggested by the Mean Value Theorem" by Professor H. H. Downing.

December 5: "A Method of Constructing the Graph of Equations in which the Variables are Separated" by Professor E. L. Rees.

December 12: "Double Points of Rational Curves" [review of O. J. Peterson's article in this MONTHLY, 1917] by Vernon G. Grove Gr.

December 19: "The Graph of $f(x)$ for Complex Numbers" [review of A. F. Frumveller's article in this MONTHLY, 1917] by Harry R. Allen Gr.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NORTH CAROLINA, Chapel Hill, N. C.

The object of this club, which was organized in October, 1916, is "to promote interest in mathematics among the students and to bring them into closer touch

[1] In honor of Professor James G. White, dean of men, at one time acting president, and for many years head of the department of mathematics in the university. He died in the summer of 1913 and Professor Boyd succeeded him as head of the department.

with the faculty in a non-class-room way." Special effort is made to interest Sophomores and Freshmen in the club.

As a rule meetings are held monthly. There are now 25 members; attendance 15–20.

Officers, 1917–18: President, Professor William Cain; vice-president, Professor Archibald Henderson; secretary-treasurer, Instructor J. W. Lasley, Jr.

The program is as follows:

October 10: Election of officers; "The Principle of Duality" by Dr. A. W. Hobbs, instructor; there was also discussion by the secretary-treasurer of several problems that had been proposed to the society for solution.

November: "A Review of the Situation in Mathematics in the High Schools of North Carolina" by Instructor William W. Rankin.

December: no meeting.

THE MATHEMATICAL CLUB OF SMITH COLLEGE, Northampton, Mass.

This club was organized in October, 1899, "to discuss matters relating to mathematics." Each year it has nine regular meetings, two or more of which are social or of a business character. Membership is limited to students of the Junior and Senior classes majoring in mathematics. Of the 35 members about one third are Juniors. Average attendance, 24.

Officers, 1917–18: President, Professor Eleanor P. Cushing; vice-president, Sarah S. Powell '18; secretary, Elizabeth Walrath '18; treasurer, Ruth Bray '18.

The program for 1917–18 is as follows:

October 22: "Mathematics Used in Actuarial Work" by Professor Ruth Wood and Threasa Boden '18.

October 30: Business meeting; election of 11 new Senior members.

November 12: "Practical Use of Mathematics in Astronomy" by Vera Gushee, demonstrator in astronomy.

December 3: Business meeting; election of 6 new Junior members.

December 10: Christmas party.

January 7: "Mathematics as applied to Domestic Arts" by Margaret McClenathan '18; "Teaching of Mathematics" by Florence Putnam '18.

TOPICS FOR CLUB PROGRAMS.

3. ARITHMETICAL PRODIGIES.

There are many instances of individuals, often very young, endowed with a power of effecting extraordinary mental calculations. Some had little or no education; the greatest of them all was stupid in everything but calculation, even including mathematics. Others attained to eminence in some field of science. Many had exceptional figure memory of either the visual or the auditory type. The rather small body of authenticated material, collected in part by commissions of scientific bodies appointed to test various prodigies, has been the

basis of considerable study by psychologists and others. A few names and facts may be recalled.

Because of exceptional calculating ability Jedediah Buxton, an Englishman of the eighteenth century, was tested by the Royal Society. George P. Bidder (1806–1878),[1] son of an English stone-mason, could multiply twelve figures by twelve figures, and perform complicated problems of compound interest.[2] He became a noted engineer and one of his great works was in connection with the designing of the Victoria Docks, London. Zerah Colburn (1804–1840), son of a Vermont farmer, seemed to possess to a remarkable degree the ability to factor numbers. In his autobiography (Springfield, Mass., 1833) occurs the following paragraph concerning an exhibition in London when 8 years of age: "It had been asserted . . . that 4,294,967,297 $(= 2^{32} + 1)$ was a prime number. . . . Euler detected the error by discovering that it was equal to $641 \times 6{,}700{,}417$. The same number was proposed to this child who found out the factors by the mere operation of his mind." In GRAVES, *Life of Sir William Rowan Hamilton*,[3] there are some interesting passages concerning Hamilton's connections with "that Zerah Colburn, the wonderful American boy who used to calculate with such astonishing rapidity." Another son of a Vermont farmer, who lived within forty miles of Colburn's birth-place, was Truman Henry Safford (1836–1901) whom Simon Newcomb has characterized as a "most wonderful genius." At the age of 6 he was able mentally to multiply four figures by four figures and to extract square and cube roots of numbers of nine and ten figures. Newcomb records that "a Committee of the American Academy of Arts and Sciences was appointed to examine him. It very justly and wisely reported that his arithmetical powers were not in themselves equal to those of some others on record, especially Zerah Colburn, but that they seemed to be the outcome of a remarkable development of the reasoning power."[4] Safford graduated from Harvard and became an astronomer. From 1876 until his death he was professor of astronomy in Williams College. Like Ampère the French physicist who was also a calculating prodigy, he had a wide range of interests and an encyclopedic memory. "Chemistry, botany, philosophy, geography as well as mathematics and astronomy, were included in his field of study." Another French prodigy was Henri Mondeux who was tested on behalf of the Academy of Sciences of the Institute of France by a committee consisting of Arago, Sturm, Cauchy and Liouville.[5]

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[1] *Dictionary of National Biography*, Vol. 5, London, 1886.

[2] He has left an excellent account of his methods in *Proceedings of the Institution of Civil Engineers*, London, 1856, vol. 15, pp. 251 ff.

[3] Volume 1, London, Longmans, 1882, pp. 77 ff. It may be noted that there is a brief reference to Colburn in De Morgan, *Budget of Paradoxes*, second edition, Chicago, Open Court, 1915, Vol. 1, p. 86.

[4] For this passage and other interesting facts see S. Newcomb, *The Reminiscences of an Astronomer*, Boston, Houghton, Mifflin, 1903, pp. 67–69. A biographical sketch may be found in *National Cyclopedia of American Biography*, vol. 13, 1906.

[5] For the report by Cauchy see Cauchy, *Œuvres complètes*, 1re série, tome 5, Paris, 1885, p. 493 f. (reprinted from *Comptes Rendus*, tome 11, pp. 820 and 952). There are biographies of Mondeux by Barbier (1841) and Jacoby (1846).

In more recent times the Italian calculator Jacques Inaudi (born 1867), who has frequently exhibited in the United States, was investigated in 1892 by a French Institute commission composed of Darboux, Poincaré, Tisserand and Charcot.[1] Périclès Diamandi (born 1868), a Greek, was tested by the same commission in 1893. The famous German mathematician Carl Friedrich Gauss (1777–1855) tells us that at the age of three he corrected calculations in connection with his father's payroll. But the most wonderful arithmetical prodigy of all time was Zacharias Dase (1824–1861) who was born in Hamburg.[2] He is known to have multiplied two 100-figure numbers in $8\frac{3}{4}$ hours and to have extracted the square root of a 100-figure number in 52 minutes. His non-mental computations included the determination of the value of π to 200 decimal places, a labor of scarcely two months;[3] the computation of a 7-place table of natural logarithms of numbers;[4] and factor tables (performed at the instigation of Gauss) for the seventh and eighth million (the latter not quite complete) and parts of the ninth and tenth million.[5]

Many facts concerning these and other prodigies, together with a discussion of psychological questions involved, and full bibliographies, may be found in the following works: E. W. SCRIPTURE, "Arithmetical Prodigies," *American Journal of Psychology*, Worcester, 1891, Vol. 4, pp. 1–59; F. D. MITCHELL, "Mathematical Prodigies," *American Journal of Psychology*, 1907, Vol. 18, pp. 61–143 (includes corrections of some of Scripture's slips); A. BINET, *Psychologie des grands calculateurs et joueurs d'échecs*, Paris, Hachette, 1894, 8 + 364 pp. (portraits of Inaudi and Diamandi); G. E. MÜLLER, *Zur Analyse der Gedächtnistätigkeit und des Vorstellungsverlaufs*, 1. und 3. Teile, Leipzig, Barth, 1911–13 (*Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, 5. und 8. Ergänzungsbände)—especially 1. Teil, pp. 177–237; W. G. SMITH, "Notes on the Special Development of Calculating Ability," pp. 60–68 of *Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition.* Edited by E. M. Horsburgh, London, Bell [1914]; and P. J. MÖBIUS,[6] *Ueber die Anlage zur Mathematik*, mit 51 Bildnissen. Leipzig, Barth, 1900—"Ueber die Rechenkünstler," pp. 66–76.

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[1] The report by Darboux together with a biographical sketch by Charcot was published in *Comptes Rendus*, 1892, vol. 114, pp. 1329–1338.

[2] There is a biographical sketch of Dase in *Allgemeine Deutsche Biographie*, Band 4, Leipzig, 1876. His name is sometimes spelled Dahse.

[3] *Crelle's Journal*, 1844, Band 27, p. 198. With the value of π on this page is printed an extract from an interesting letter written by S. Strasznicky of Vienna.

[4] Z. Dase, *Tafel der natürlichen Logarithmen*, Vienna, 1850 (reprinted from *Annals* of the Vienna Observatory for 1851). *Cf.* J. W. L. Glaisher's article on "Table, Mathematical" in the eleventh edition of the *Encyclopædia Britannica*.

[5] The tables for the seventh million were printed at Hamburg in 1862 and the tables for the eighth and ninth millions, completed by H. Rosenberg, were published at the same place in 1863 and 1865 respectively. The tenth million by Dase and Rosenberg has not been published but The Carnegie Institution of Washington issued in 1909 a *Factor Table for the First Ten Millions* by D. N. Lehmer. The introduction to this table lists only 7 errors in Dase's seventh million table (1862) but over 130 in the volumes in which Rosenberg collaborated.

[6] A doctor; grandson of the well-known astronomer and mathematician.

In H. A. BRUCE, "Lightning Calculators, a study in the psychology of harnessing the subconscious," *McClure's Magazine*, 1912, Vol. 39, pp. 586–596, there are portraits of W. J. Sidis, G. Bidder, H. T. Safford, Gauss, Griffith (an Indiana calculator) and Inaudi. Another brief sketch is G. A. MILLER, "Mathematical Prodigies," *Science*, new series, Vol. 26, pp. 628–30, 1907 (also *Scientific American Supplement*, Vol. 65, p. 51, 1908).

4. PTOLEMY'S THEOREM AND FORMULÆ OF TRIGONOMETRY.

Ptolemy of Alexandria, celebrated mathematician, astronomer and geographer, was a native of Egypt and flourished about the middle of the second century of the Christian era.[1] His great treatise on astronomy is usually known as the *Almagest*—a name derived from an Arabic form of its title. The work is divided into thirteen books.[2] In discussing matters pertaining to plane and spherical trigonometry in the first book Ptolemy introduces as a lemma what is now known as Ptolemy's Theorem: "The rectangle contained by the diagonals of any quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by the pairs of opposite sides" (Heiberg's edition, volume 1, pages 36–7). An English translation of Ptolemy's proof of his theorem is given in T. L. HEATH, *The Thirteen Books of Euclid's Elements*, Vol. 2, Cambridge University Press, 1908, pages 225–6.

From this theorem Ptolemy readily derived the equivalent of the formulæ:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

and

$$\sin\frac{\alpha}{2} = \left(\frac{1-\cos\alpha}{2}\right)^{1/2}.$$

Of French and English discussions of similar applications of the theorem the following may be noted: L. N. M. CARNOT, (*a*) *De la corrélation des figures de géométrie*, Paris, 1801, pp. 83f., 95f.; (*b*) *Géométrie de position*, Paris, 1803, pp. 151ff.; W. A. WHITWORTH, *Mathematical Gazette*, London, 1904, Vol. 3, pp. 6–7; J. JUHEL-RÉNOY, *Bulletin des sciences mathématiques et physiques élémentaires*, Paris, 1908, Vol. 13, p. 113; A. G. BURGESS, "Ptolemy's Theorem and Certain Trigonometrical Formulæ," *Mathematical Notes*, Edinburgh, 1910, No. 4, pp. 42–3; and A. BABBITT, *School Science and Mathematics*, 1917, Vol. 17, p. 784.

Among the many proofs of Ptolemy's Theorem reference may be given to two others: (*a*) by inversion in F. G. M., *Exercices de géométrie*, 5e éd., Paris, Gigord, 1912, pp. 104–5; (*b*) by means of the anharmonic relation

$$E(BACD) + E(BCAD) = 1,$$

[1] *Cf.* the article on Claudius Ptolemy in *Encyclopædia Britannica*, eleventh edition.

[2] The best text has been edited by J. L. Heiberg: *Syntaxis Mathematica*, 2 vols., Leipzig, Teubner, 1898–1903. The catalogue of stars contained in the seventh and eighth books was published by the Carnegie Institution of Washington in 1915. For a commentary on the *Almagest* see Delambre, *Histoire de l'astronomie ancienne*, tome 2, Paris, 1817, pp. 67–410 and 36–44.

where A, B, C, D are the vertices of the inscribed quadrilateral, and E any other point on the circle (J. JUHEL-RÉNOY, *Nouvelles annales de mathématiques*, 1906, Vol. 65, p. 12; *cf.* J. L. COOLIDGE, *A Treatise on the Circle and on the Sphere*, Oxford, Clarendon Press, 1916, p. 315).

By taking account of the directions of line segments, Moebius arrived at the generalization of Ptolemy's Theorem:

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0,$$

where A, B, C, D may be arranged in any way whatever on the circle.[1] The discovery that this relation is true when the points are collinear is usually attributed to Euler, but the result was known to a tenth-century Arab, El-Sābī (*Bibliotheca Mathematica*, 3. Reihe, Vol. 8, 1907–8, p. 24).

Carnot generalized Ptolemy's Theorem to apply to the case of an inscribed polygon (*Géométrie de position*, p. 273). For John Casey's generalization (1866) see his *Sequel to Euclid*, London, Longmans, 1900, pp. 103–4, and J. L. COOLIDGE, *A Treatise on the Circle*, etc., p. 38.

Amplification of some of the ideas presented in sources indicated above may be found in: J. L. COOLIDGE, *A Treatise on the Circle*, etc., p. 138; X. ANTOMARI, "Relation entre les distances mutuelles: 1° de quatre points situés sur un même cercle; 2° de cinq points situés sur une même sphère," *Nouvelles annales de mathématiques*, 1882, tome 41, pp. 462–64; A. KRAHÉ, "Sur les relations de Ptolémée et de Feuerbach," *Mathesis*, 1898, tome 18, pp. 83–84; W. F. MEYER, "Über den Ptolemäischen Satz," *Archiv der Mathematik und Physik*, 1903, 3. Reihe, Band 7, pp. 1–15; M. STUYVAERT, "Conséquences diverses d'une formule d'algèbre; leurs interprétations géométriques," *L'Enseignement mathématique*, 1906, tome 8, pp. 282–290; and MERRIFIELD, *Proceedings of the London Mathematical Society*, 1881, Vol. 12, p. 214.

5. PAPER FOLDING.

In 1893 an Indian, T. Sundara Row, published a little volume of 120 pages[2] setting forth, very ingeniously, many exercises in paper folding which constitute an interesting chapter in mathematical recreations. There are constructions for: (*a*) the regular pentagon, hexagon, octagon, nonagon, decagon, dodecagon, and pentedecagon; (*b*) points on the conic sections and on such curves as the cissoid, the conchoid of Nicomedes, the witch of Agnesi, the cubical parabola, the lemniscate, and the cycloid. Many theorems of Euclidean and projective geometry are also discussed.

Apparently without knowledge of Sundara Row's work H. Wiener gave also in 1893 an account of the construction of the regular convex polyhedra by paper

[1] A. F. MOEBIUS, "Die Theorie der Kreisverwandtschaft, etc.," 1855; *Gesammelte Werke*, Band 2, Leipzig, 1885, pp. 280, 307.

[2] *Geometrical Exercises in Paper Folding.* By T. Sundara Row . . . Madras, Addison and Co. A new edition with slight modifications of the proofs and the addition of a "considerable number of half-tone reproductions of actual photographs instead of the line-drawings of the original" was brought out by W. W. Beman and D. E. Smith in 1901 (Chicago, Open Court).

folding.[1] When it is recalled that for over a century the simple method of constructing such polyhedra has been given in text-books, it appears that Wiener's additional contribution consisted almost wholly in constructing, by paper folding, equilateral triangles, squares, and pentagons—a very small portion of Sundara Row's discussion. This fact is not made clear in Klein's remark on page 42 of the English edition of his *Famous Problems in Elementary Geometry*, Boston, Ginn, 1897.

Finally, reference may be given to a note by Fitz Patrick on "La géométrie par le pliage et découpage du papier" in W. W. R. BALL, *Récréations mathématiques et problèmes des temps anciens et modernes*, troisième partie, Paris, Hermann, 1909, pp. 341–360; to *Messenger of Mathematics*, 1905, Vol. 34, pp. 142–3; and to *L'Education Mathématique*.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Dr. J. B. ROSENBACH has been appointed instructor in mathematics at the University of New Mexico.

Dr. GOLDIE P. HORTON has been appointed instructor in mathematics at the University of Texas.

Dr. G. H. LIGHT, of the University of Colorado, has been promoted to an assistant professorship of mathematics.

Dr. E. W. PONZER, assistant professor of applied mathematics at Leland Stanford University, died on December 20, 1917, as a result of an accidental gunshot wound.

Mr. J. J. TANZOLA, until recently an instructor in mathematics at the U. S. Naval Academy, is now a private with the 305th Machine Gun Battalion, Company C, at Camp Upton, Long Island.

Mr. FREDERICK WOOD, instructor in mathematics at the University of Wisconsin and a charter member of the Association, is now a lieutenant with the 328th Field Artillery at Camp Custer, Michigan.

Dr. J. E. DAVIS, instructor in mathematics at Pennsylvania State College, is enrolled in the 313th Infantry at Camp Meade, Md.

The University of California has been, since May 21, 1917, conducting a school of military aëronautics, in which about five hundred cadets are being trained in an eight weeks' course in flying. Professor B. M. WOODS, of the department of mathematics, a charter member of the Association, is directing the instruction in this school.

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[1] *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente.* Nachtrag. Herausgegeben von W. Dyck. München, Wolf, 1893, pp. 52–54.

Professor A. O. LEUSCHNER, of the University of California, has been delegated to administer instruction in the navigation schools conducted by the U. S. Shipping Board on the Pacific coast.

On account of conditions due to the war, it has been decided by Harvard University that no appointments be made to the Benjamin Peirce instructorships in mathematics during the present year.

Professor J. N. VAN DER VRIES, of the University of Kansas and a member of the Council of the Association, has been chosen as a member of the field division of the U. S. Chamber of Commerce. He has been given a year's leave of absence in order to carry out certain work in connection with the organization of war service committees, which are to take the place of the old national defense council committees and thus to furnish a point of direct contact between specific industries and the government.

Professor HARRIS HANCOCK, of the University of Cincinnati, was elected treasurer of the American Association of University Professors at its Chicago meeting during the holidays.

At the Pittsburgh meeting of the American Association for the Advancement of Science, Assistant Professor G. D. BIRKHOFF, of Harvard University, was elected vice-president of Section A (Mathematics and Astronomy); Professor F. R. MOULTON, of the University of Chicago, was reëlected secretary of the Section.

The initial number of Vol. XL, of the *American Journal of Mathematics*, appeared in January with six mathematical contributions as follows: "Flat-sphere geometry," by J. EISLAND; "Irrational involutions on algebraic curves," by J. V. DE PORTE; "The set of eight self-associated points in space," by J. R. MUSSELMAN; "Associate minimal surfaces," by J. K. WHITTEMORE; "On integral invariants," by F. W. REED; "Fundamental regions for certain finite groups in S_4," by H. F. PRICE.

The *Science Reports* of the Tôhoku Imperial University, November 1917, contains an interesting mathematical discussion by K. TERAZAWA on the "Oscillations of the deep-sea surface caused by a local disturbance." The present paper is a continuation of the writer's former discussion of the problem, *Proc. Roy. Soc.*, London, 1915, and is intended to complete the discussion of the problem of central oscillations with the initial conditions: (1) when an initial displacement of the free surface is given, without initial velocity; (2) when an initial impulse is applied on the surface, without initial surface displacement. The same issue of *Science Reports* also contains a paper by T. KUBOTA on "An application of binary quadratic forms to geometry."

The concluding number of Vol. 12, *Tôhoku Mathematical Journal*, edited by T. HAYASHI, Sendai, Japan, recently appeared containing the following papers: "On the theory of representation of surfaces," by K. OGURA; "A theorem on

limits," by S. NARUMI; "On generalized Toeplitz theorems on limits and their applications," by T. KOJIMA. This twelfth volume of the *Journal* contains 334 pages, consisting, for the most part, of original papers written in the English language.

A continuation of the discussion of "Mathematics in secondary schools," following the investigation conducted by a committee of the Chicago Mathematics Club, appears in *School and Society*, January 12, 1918, by Professor C. N. MOORE, of the University of Cincinnati. This paper is devoted chiefly to a criticism of Professor SNEDDEN's discussion of the same subject, in *School and Society*, December 1, 1917. "Mathematics as a test of mental efficiency," by Professor R. E. MORITZ of the University of Washington, appears in the January 12 issue of *School and Society*.

"A glimpse at early colonial algebra" is the subject of a most interesting historical paper by Professor D. E. SMITH, of Teachers College, Columbia University, in *School and Society*, January 5, 1918. Professor SMITH writes concerning the early manuscript copies on mathematics found in the collections of colonial libraries, and in particular he gives a description of a manuscript copy on algebra written by SAMUEL LANGDON under the direction of ISAAC GREENWOOD, Hollis Professor of Mathematics at Harvard University, 1727–1738. At the time of writing the manuscript on algebra, LANGDON was fifteen years old; he later became president of Harvard University, serving during the American Revolution.

On February 3, 1918, Dr. Ellery W. Davis, dean of the college of literature, science and arts of the University of Nebraska and professor of mathematics, died of pneumonia after a brief illness. Dr. Davis was a charter member of the Mathematical Association of America. He was present at the recent Chicago meeting of the association and presided at one of the sessions.

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RABBI BEN EZRA AND THE HINDU-ARABIC PROBLEM.

By DAVID EUGENE SMITH and JEKUTHIAL GINSBURG.

When Browning put in the mouth of Rabbi ben Ezra the words

"The Future I may face now I have proved the Past,"

he wrote better than he knew, for no scholar of the twelfth century had proved the Past more thoroughly than he, and few could face the Future with greater confidence.[1] Born, very likely at Toledo, between 1093 and 1096,[2] he became known as one of the most learned men of his time, and was the author of numerous books, a traveled scholar, a poet, and a man of great influence. He died in 1167, probably either in Rouen or in Rome.

Of the many lines of research pursued by Rabbi ben Ezra, one of the most interesting to students of the history of mathematics is that relating to the introduction of Hindu astronomy and computation into the Arabian civilization. Of that remarkable activity of the seventh century which resulted in the amalgamation of numerous semi-nomadic tribes into one mighty empire we have abundant knowledge; of the opening of the golden age of Mohammedan civilization and of its development under the caliphs of Bagdad in the eighth and ninth centuries we have well-authenticated records; of the influx of the Greek civilization through the translation of the classics of Alexandria and Athens we have the witness of a large number of manuscripts in the great collections of Europe and

[1] In the text of this article, the name Rabbi ben Ezra will be used because it is familiar to English readers. Properly, the name should be written Abraham ibn Ezra, although it often appears in the Latin translations as Abraham Judæus. In the notes we shall use the form Ibn Ezra.

[2] For a discussion of the matter, see Steinschneider, "Abraham Ibn Esra" in the *Abhandlungen zur Geschichte der Mathematik*, III, 59, Leipzig, 1880.